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Regular and exact completions

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Abstract

The regular and exact completions of categories with *weak* limits are proved to exist and to be determined by an appropriate universal property. Several examples are discussed, and in particular the class of examples given by categories monadic over a power of Set: any such a category is in fact the exact completion of the full subcategory of free algebras. Applications to Grothendieck toposes and geometric morphisms, and to epireflective hulls are also discussed. © 1998 Elsevier Science B.V.

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1. Introduction

The notions of a regular and of an exact category are among the most interesting notions studied in category theory. In fact, several important mathematical situations can be axiomatized in categorical terms as regular or exact categories satisfying some typical axioms. Let us recall that a category is *regular* (see [1]) when

(i) it is left exact,
(ii) each arrow can be factored as a regular epi followed by a mono,
(iii) regular epis are pullback stable,

and is *exact* when moreover

(iv) equivalence relations are effective (i.e. kernel pairs).

Exact functors between regular or exact categories are left exact functors which preserve regular epimorphisms.

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For instance, *small* regular categories are the basis for an invariant definition of first-order (intuitionistic) theories (see [11, 19]). All monadic categories over a power of Set, and in particular algebraic categories, are exact. Grothendieck toposes are exact categories, and the only difference with algebraic categories is the behaviour of coproducts and the projectivity of the generators or, in other words, the facts that a Grothendieck topos has disjoint and universal sums and does not necessarily have "*enough projectives*". Other examples are: the dual categories of toposes – they always are exact categories and have enough projectives; the category of topological groups, which is regular, but not exact; the category of compact groups, which is exact. Finally, a large class of examples is given by abelian categories: an abelian category is an exact category satisfying moreover the typical axiom which holds in module categories, i.e. the set of homomorphisms between two objects is an abelian group and left and right compositions with any given map are group homomorphisms.

As it is always the case in mathematics, when a new relevant structure emerges and begins to be studied as such, an immediate question is the study of the "*free*" such structures. Of course, "free" refers to a given forgetful functor, and in the case of regular and exact categories there are several such forgetful functors whose corresponding free functor (left adjoint to the forgetful) should be investigated, namely – in increasing order of complexity – the ones into graphs, categories and left exact categories. They all exist for general reasons, and an explicit description of the last has been given in [7].

It has recently been observed that in the description of the free exact category over a left exact one given in [7], the fact that the starting category has limits is never fully used, and that in fact only the *existence* property in the definition of a limit is used. In other words, the construction of the free exact category over a left exact one only uses "*weak limits*", that is, starting with a category with weak limits, the same construction gives a category *with honest limits* and which is exact. A bit different is the case of regular categories: the construction of the free regular category over a left exact one can be extended to categories with weak limits, but the construction should be slightly modified to get products.

The natural question then arises to investigate the universal property of the constructions for categories with weak limits, and the surprise is that the wished one, that is adjointness to the fortgetful into the 2-category of categories with weak limits and weak limits preserving functors, does *not* hold. More, whatever class of functors we take between categories with weak limits, provided it is stable by composition and it is such that the construction of the "free" exact category remains functorial, our construction will never give adjointness. However, it has been soon realized that there is another universal property, different from adjointness, which holds and determines the construction up to equivalences – a quite unusual phenomenon in Category Theory, whose only ancestor we know is Freyd's representation theorems in abelian categories [10].

This paper is devoted to the study of such constructions, both for regular and exact categories, and of their universal property. The paper is organized as follows.

Section 2 is devoted to the explicit descriptions of the regular and exact completions of categories with finite weak limits, and to the characterization of categories which occur as such. We find the characterization of exact completions of categories with finite weak limits remarkable: an exact category is the exact completion, necessarily of its full subcategory of (regular) projectives, if and only if it has *enough projectives* – an axiom which, in the linear case, is the basis for homological algebra, and which for algebraic categories tells us that they are *the exact completions of the full subcategories of free algebras*.

Section 3 deals with the universal property of the constructions: the key definition is that of a *left covering functor*, that is of a functor from a category with weak limits into a category with honest limits such that for any diagram, the canonical comparison from the image of any weak limit of the diagram to a honest limit of the image of the diagram, is a strong epimorphism. When the codomain category is regular or exact, this is precisely the class of functors which extend to exact functors from the completions, in an essentially unique way. We end up the section investigating the cocompleteness of the exact completions, a situation which often occur in the examples we have in mind.

Finally, in Section 4 we discuss examples and applications. We already mentioned the case of algebraic categories and, more generally, of monadic categories over a power of Set: they all are exact completions of their full subcategories of projectives. This fact gives various characterization theorems, some already known and some new – notably the one characterizing localizations of algebraic categories – discovered in the meantime by the second named author (see [23]), but which we recall here for completeness and to emphasize the role of the theory developed in the first two sections in clarifying and unifying this matter. We also show that this role is played in Topos Theory, by giving new simple proofs of known facts concerning toposes as localizations of presheaf categories and concerning geometric morphisms. We end up extending our theory of regular and exact completions to categories with *all* small weak limits, to cover a wider range of examples of regular completions: we show that in the monadic case the so-called "epireflective hull" of the full subcategory of the projectives is in fact also its infinitary regular completion, and hence enjoys an appropriate universal property. This applies, for instance, to the category of Stone spaces, to the dual categories of topological spaces and of sober spaces. We would like to mention a possible application which has been *not* yet explored, but which we feel it should: several examples of categories with weak limits which occur in "nature" arise in homotopy theory, as the homotopy categories of various categories of interest, such as topological spaces, groupoids, simplicial sets. Are their regular or exact completions of some help in homotopy theory? We do not know.

A bit of history: at the 1991 Category Theory Conference in Montreal, the first named author telling about the fact that he had just observed, during a visit of Makkai, the existence of the regular and exact completions of categories with weak limits, asked for the universal property. The second named author then discovered it, and a quite complete treatment of it has been the subject matter of his 1994 doctoral dissertation [21] at the University of Louvain, under the supervison of Borceux, whom we thank for some useful discussions and for having arranged visits of the first named author to Louvain-la-Neuve. In the meantime, Hu independently discovered the same universal property, and in [14] gave an elegant proof of it using duality theory. Unfortunately, his theory applies so far only to exact categories and to *small* ones, although it seems quite reasonable that it can be extended to locally small ones; moreover, it is not *elementary*, and so it does not eliminate the need of an explicit elementary description. Hence, we decided to join to make available our theory with all extensions, applications and examples we have discovered so far.

2. Regular and exact completions

2.1. Weak limits and pseudo-equivalence relations

Recall that a weak limit is defined as the usual notion of limit, except that one requires only the existence of a factorization and not the uniqueness. A first consequence of this fact is that a functor can admit several non-isomorphic weak limits; for example, in the category Set of sets, each non-empty set is a weak terminal object.

The following is the expected condition for the existence of finite weak limits; we give some details because the way to build up a weak limit from weak products and weak equalizers will be useful in other sections.

Proposition 1. *The existence of weak binary products and weak equalizers implies the existence of all weak finite non-empty limits.*

Proof. The existence of weak pullbacks follows as in the case of usual limits and in the same way binary implies finite. Now, if $\mathscr{L} : \mathscr{D} \to \mathbb{C}$ is a functor defined on a finite category $\mathscr{D}$, let us consider a weak product indexed over all the objects $D \in \mathscr{D}_0$ with the corresponding projections

$$\pi_D : \left(\prod_{D \in \mathscr{D}_0} \mathscr{L}D \right) \to \mathscr{L}D;$$

for each arrow $d : D \to D'$, consider the two parallel arrows in $\mathbb{C}$

$$\prod_{D \in \mathscr{D}_0} \mathscr{L}D \underset{\mathscr{L}d\pi_D}{\overset{\pi_{D'}}{\rightrightarrows}} \mathscr{L}D'$$

and let

$$E_d \xrightarrow{e_d} \prod_{D \in \mathscr{D}_0} \mathscr{L}D \underset{\mathscr{L}d\pi_D}{\overset{\pi_{D'}}{\rightrightarrows}} \mathscr{L}D'$$

be a weak equalizer of them. Since weak pullbacks exist as in the ordinary case, there exists a weak limit $(E \xrightarrow{e'_d} E_d)_{d \in \mathscr{D}_1}$ of the diagram

$$\left(E_d \xrightarrow{e_d} \prod_{D \in \mathscr{D}_0} \mathscr{L}D \right)_{d \in \mathscr{D}_1}.$$

It is now easy to check that the cone

$$\left(E \xrightarrow{e'_d} E_d \xrightarrow{e_d} \prod_{D \in \mathscr{D}_0} \mathscr{L}D \xrightarrow{\pi_D} \mathscr{L}D \right)_{D \in \mathscr{D}_0}$$

is a weak limit of $\mathscr{L}$. □

Observe that the statement that "pullbacks and terminal object suffice for finite limits" is not anymore true for the weak notion. We will call a category $\mathbb{C}$ "*weakly lex*" if, for every functor $\mathscr{L} : \mathscr{D} \to \mathbb{C}$ defined on a finite category $\mathscr{D}$, there exists a weak limit of $\mathscr{L}$, which we will denote by "wlim $\mathscr{L}$".

Recall that a *regular epi* is a map which is a coequalizer, and that an object P is (*regular*) *projective* when the covariant hom-functor preserves regular epis.

Definition 2. Let $\mathbb{C}$ be a category and $\mathbb{P}$ be a full subcategory of $\mathbb{C}$; we say that $\mathbb{P}$ is a *"projective cover of* $\mathbb{C}$" if the following two conditions are satisfied:
- each object of $\mathbb{P}$ is regular projective in $\mathbb{C}$,
- for each object X of $\mathbb{C}$ there exists a $\mathbb{P}$-cover of X, that is an object P of $\mathbb{P}$ and a regular epi $P \to X$.

Of course, a category $\mathbb{C}$ admits a projective cover if and only if it has "*enough projectives*", i.e. every object is the codomain of a regular epi whose domain is projective. Elsewhere a projective cover is called a "resolving set of projectives" (see, for example, [10]). The relation between two projective covers of the same category is clarified in the following proposition.

Proposition 3. *Let* $\mathbb{P}_1$ *and* $\mathbb{P}_2$ *be two projective covers of a category* $\mathbb{C}$; *the splitting of idempotents of* $\mathbb{P}_1$ *is equivalent to those of* $\mathbb{P}_2$; *in particular, if the idempotents split in* $\mathbb{C}$, *the splitting of idempotents of* $\mathbb{P}$ *is equivalent to the full subcategory of all the projective objects of* $\mathbb{C}$.

The relation between weak limits and projective covers is given in the following proposition.

Proposition 4. *Let* $\mathscr{L} : \mathscr{D} \to \mathbb{C}$ *be a functor defined on a finite category* $\mathscr{D}$ *and suppose that it can be factored as*

$$\begin{array}{ccc} \mathscr{D} & \xrightarrow{\mathscr{L}} & \mathbb{C} \\ & {}_{\mathscr{L}'}\searrow \quad \nearrow & \\ & \mathbb{P} & \end{array}$$

where $\mathbb{P}$ *is a projective cover of* $\mathbb{C}$ *and* $\mathbb{P} \to \mathbb{C}$ *is the inclusion*; *if there exists wlim* $\mathscr{L}$, *then there exists also wlim* $\mathscr{L}'$. *In particular if* $\mathbb{C}$ *is weakly lex, the same holds for* $\mathbb{P}$.

Proof. Let $(\pi_D : L \to \mathscr{L}D)_{D\in\mathscr{D}_0}$ be a weak limit of $\mathscr{L}$ and consider a $\mathbb{P}$-cover $p : P \to L$ of L; then $(\pi_D p : P \to \mathscr{L}D)_{D\in\mathscr{D}_0}$ is a weak limit of $\mathscr{L}'$: for any cone $(\tau_D : Q \to \mathscr{L}D)_{D\in\mathscr{D}_0}$ on $\mathscr{L}$ with $Q \in \mathbb{P}$, the factorization $\tau : Q \to L$ can be lifted to a factorization $\tau' : Q \to P$ because Q is projective. □

A large class of examples occurring in nature of weakly lex categories is given by *exact categories with enough projectives*: if $\mathbb{P}$ is a projective cover of such a category, then $\mathbb{P}$ *has all finite weak limits*, by Proposition 4.

Definition 5. A category $\mathbb{E}$ is called "regular" (see [1]) when

(1) it is left exact;

(2) every effective equivalence relation (i.e. a kernel pair) has a coequalizer;

(3) pullbacks of regular epis are regular epis.

$\mathbb{E}$ is called "*exact*" when is regular and

(4) every equivalence relation is effective.

Functors between regular or exact categories are called "*exact*" when they preserve finite limits and regular epis.

It is now quite an easy and instructive exercise to show that regular categories admit a (regular epi)-(mono) factorization, and that they have all the properties which allow the "calculus of relations". In fact, in the presence of axiom (3), axiom (2) can be equivalently stated as

(2′) every map has a (regular epi)-(mono) factorization;

which is a bit redundant, but more illuminating. Also, a basic fact which illuminates the role of regular categories in categorical logic, is Joyal's theorem that regular categories defined as those satisfying (1), (2′) and (3), where regular epis are replaced by "*strong epis*", i.e. by those maps which do not factor through any proper subobject, agrees with the one given here, because in such a category every strong epi is in fact regular. The reader may consult [1, 4, 11, 17] for a more detailed account on regular and exact categories. Let us only mention here a large class of examples of exact categories, namely *monadic categories over (a power of) sets*, in particular *algebraic categories* and *presheaf toposes*, which always have enough projectives, and their *localizations*, which rarely have enough projectives.

As we said, Proposition 4 implies that any projective cover of an exact category has all finite weak limits: just take projective covers of the honest limits. Our goal is to show that in fact this class of examples is the class of *all possible examples*, in the sense that *any weakly lex category appears as a projective cover of an exact category*. To have a better understanding of the construction, let us consider an exact category $\mathbb{E}$ with a projective cover $\mathbb{P}$, and let us show *how we can reconstruct* $\mathbb{E}$ *out of* $\mathbb{P}$: let A be an object of $\mathbb{E}$; choosing a $\mathbb{P}$-cover $a : X \to A$ of A, consider its kernel pair a_0, a_1, and take a $\mathbb{P}$-cover R of its domain; we obtain a parallel pair $r_0, r_1 : R \rightrightarrows X$ in $\mathbb{P}$, which still has a as a coequalizer and which is a "*pseudo-equivalence relation*" in the weakly lex category $\mathbb{P}$, in the sense of the following:

Definition 6. (1) In a category $\mathbb{C}$, a pseudo-relation on an object X is a pair of parallel arrows $r_0, r_1 : R \rightrightarrows X$; the pseudo-relation is a relation if r_0 and r_1 are jointly monic;
(2) the pseudo-relation $r_0, r_1 : R \rightrightarrows X$ is
– "reflexive", if there exists an arrow $r_R : X \to R$ such that

$$r_0 r_R = 1_X = r_1 r_R,$$

– "symmetric", if there exists an arrow $s_R : R \to R$ such that

$$r_0 s_R = r_1, \qquad r_1 s_R = r_0,$$

– "transitive" if there exists a weak pullback,

$$\begin{array}{ccc} P & \xrightarrow{l_0} & R \\ {\scriptstyle l_1}\downarrow & & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow[r_0]{} & X \end{array}$$

and an arrow $t_R : P \to R$ such that $r_0 l_0 = r_0 t_R$ and $r_1 l_1 = r_1 t_R$.

Let us remark that the transitivity of a pseudo-relation $r_0, r_1 : R \rightrightarrows X$ does not depend on the choice of the weak pullback of r_0 and r_1. Of course, a "*pseudo-equivalence relation*" is a pseudo-relation which is reflexive, symmetric and transitive.

Now, in the case of the weakly lex category given by a projective cover $\mathbb{P}$ of an exact category $\mathbb{E}$, *we can recover all the maps in* $\mathbb{E}$ between two objects A and B by means of the maps in $\mathbb{P}$ only, as follows: choose two $\mathbb{P}$-covers $a : X \to A$ and $b : Y \to B$ of A and B, respectively, and consider the pseudo-equivalence relations associated with them as described above, say $r_0, r_1 : R \rightrightarrows X$ and $s_0, s_1 : S \rightrightarrows Y$; then maps $A \to B$ in $\mathbb{E}$ are in bijection with equivalence classes of pairs of maps $\bar{f} : R \to S$ and $f : X \to Y$ in $\mathbb{P}$ making commutative the two obvious squares, two such pairs $f,\bar{f}$ and $g,\bar{g}$ being equivalent if there exists an "*half-homotopy*" between them, i.e. a map $\Sigma : X \to S$, whose compositions with the two structural maps are f and g, respectively. For, given an arrow $\alpha : A \to B$ in $\mathbb{E}$, since X is projective and b is a regular epi, there exists $f : X \to Y$ such that $\alpha a = bf$; this implies that $bfr_0 = bfr_1$ so that there exists $\tilde{f} : R \to M$ such that $fr_0 = b_0\tilde{f}$ and $fr_1 = b_1\tilde{f}$ (remember that $b_0, b_1 : M \rightrightarrows Y$ denotes the kernel pair of b); but also R is projective, so that there exists $\bar{f} : R \to S$ such that $s\bar{f} = \tilde{f}$, where $s : S \to M$ is a $\mathbb{P}$-cover of M. Observe that $s_0\bar{f} = fr_0$ and $s_1\bar{f} = fr_1$, and that α is the unique map induced by this pair on the coequalizers.

If g, $\bar{g}$ is another pair of arrows making the three squares commutative,

$$\begin{array}{ccccc} R & \overset{r_0}{\underset{r_1}{\rightrightarrows}} & X & \xrightarrow{a} & A \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \alpha} \\ S & \overset{s_0}{\underset{s_1}{\rightrightarrows}} & Y & \xrightarrow[b]{} & B \end{array}$$

then $bf = bg$; if we consider the monic part $b_0, b_1 : M \rightrightarrows Y$ of the regular epi-jointly monic factorization of $s_0, s_1 : S \rightrightarrows Y$, we know that $b_0, b_1 : M \rightrightarrows Y$ is the kernel pair of its coequalizer b. The last equation implies then that there exists $\sigma : X \to M$ such that $b_0\sigma = f$ and $b_1\sigma = g$; but X is projective and s is a regular epi, so there exists $\Sigma : X \to S$ such that $s\Sigma = \sigma$; this implies that $s_0\Sigma = f$ and $s_1\Sigma = g$, that is Σ is an half-homotopy, as required.

The above argument suggests to consider the category $\mathbb{P}_{\text{ex}}$ whose objects are pseudo-equivalence relations in $\mathbb{P}$ and whose maps are equivalence classes of pairs of maps as described above. We can then choose a functor

$$\mathbb{P}_{\text{ex}} \to \mathbb{E}$$

sending each pseudo-equivalence relation in $\mathbb{P}$ into a coequalizer in $\mathbb{E}$, and show that it is in fact an *equivalence*, since it is full and faithful, as we have just shown, and it is representative, because $\mathbb{P}$ is a projective cover of $\mathbb{E}$.

The remarkable fact we will discuss in this section is that the way to reconstruct an exact category $\mathbb{E}$ from a projective cover $\mathbb{P}$ we just described, can be abstractly repeated starting from any weakly lex category $\mathbb{C}$, giving as a result a category with honest limits, which is also an *exact category* having $\mathbb{C}$ as a projective cover, which we will call the "*exact completion*" *of* $\mathbb{C}$. The way we will proceed is first to show that starting from a weakly lex category $\mathbb{C}$ we can construct a *regular category* having $\mathbb{C}$ as a projective cover, which we will call the "*regular completion*" *of* $\mathbb{C}$; then by applying a known construction, *the* "*exact completion of a regular category*", we will get the exact completion of the weakly lex category $\mathbb{C}$.

2.2. *The regular completion of a weakly lex category*

If $\mathbb{E}$ is only regular with enough projectives, the subcategory $\mathbb{P}$ of projectives still has weak limits. Let us now discuss under what conditions we can recover the regular category $\mathbb{E}$ from its full subcategory of projectives, provided they are enough. Since $\mathbb{E}$ is not exact, the argument that pseudo-equivalence relations have coequalizer does not work anymore, since now only *effective* equivalence relations have coequalizer; however, if also the property that "*every object of* $\mathbb{E}$ *can be embedded in a projective*" holds, then we can argue as follows: given an object A of $\mathbb{E}$, we can consider a factorization

$$a : P_1 \to A \hookrightarrow P_0,$$

where P_0 and P_1 are projectives and observe that, using the notations of Section 2.1, the pseudo-equivalence relation in $\mathbb{P}$ associated to the projective cover $P_1 \to A$

$$a_0, a_1 : R \rightrightarrows P_1$$

is now a weak kernel of the map a in the weakly lex category $\mathbb{P}$. By repeating the argument in Section 2.1, if b is another such factorization of an object B,

$$b : Q_1 \to B \hookrightarrow Q_0\,,$$

then maps $A \to B$ in $\mathbb{E}$ correspond bijectively to equivalence classes of maps between the pseudo-equivalence relations associated, as described in Section 2.1. Observe that in this case, we can simplify the description of maps between the pseudo-equivalence relations using that now are weak kernels in $\mathbb{P}$, simply by saying that such a map is a map $\alpha : P_1 \to Q_1$ such that $b\alpha a_0 = b\alpha a_1$. The half-homotopy equivalence relation can be simplified too, because two such maps α and α' are equivalent with respect to the half-homotopy equivalence relation if and only if $b\alpha = b\alpha'$. This suggests to consider the category $\mathbb{P}_{\text{reg}}$ whose objects are arrows $a : P_1 \to P_0$ of $\mathbb{P}$, and whose maps $a \to b$ are equivalence classes of maps f from the domain of a to the domain of b such that $bfa_0 = bfa_1$, where a_0, a_1 is a weak kernel pair of a in $\mathbb{P}$, and the equivalence relation is the half-homotopy relation. We can then choose a functor

$$\mathbb{P}_{\text{reg}} \to \mathbb{E}$$

sending each map between projectives into its *regular image* in $\mathbb{E}$, and show that it is an *equivalence*, because every object of $\mathbb{E}$ can be embedded in a projective.

We can now try to repeat the construction just described, starting with any weakly lex category $\mathbb{C}$, but there is a problem: we cannot show that $\mathbb{C}_{\text{reg}}$ has *finite products.* To show that it has finite products we have to modify the construction as follows.

Definition 7. Let $\mathbb{C}$ be a weakly lex category; we define a new category $\mathbb{C}_{\text{reg}}$ (the "regular completion of $\mathbb{C}$") as follows:
- *objects*: An object of $\mathbb{C}_{\text{reg}}$ is a finite family of arrows $(f_i : X \to X_i)_I$ in $\mathbb{C}$;
- *arrows*: An arrow $(f_i : X \to X_i)_I \to (g_j : Y \to Y_j)_J$ is an arrow $\alpha : X \to Y$ of $\mathbb{C}$ such that $g_j \alpha x_0 = g_j \alpha x_1$, for all $j \in J$, where $x_0, x_1 : \overline{X} \rightrightarrows X$ is a weak joint kernel of the family $(f_i : X \to X_i)_I$ (i.e. it is weakly universal with respect to the property $f_i x_0 = f_i x_1$, for all $i \in I$):

$$\begin{array}{ccc} \overline{X} & & \\ {\scriptstyle x_0}\downarrow\downarrow{\scriptstyle x_1} & & \\ X & \xrightarrow{\alpha} & Y \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle g_j} \\ X_i & & Y_j \end{array}$$

  Two arrows of this kind $\alpha : X \to Y$ and $\alpha' : X \to Y$ are declared to be equivalent if $g_j \alpha = g_j \alpha'$, for all $j \in J$;
- *composition and identities*: the obvious ones.

It is straightforward to verify that the previous data define a category. We will use the notation $[\alpha] : (f_i) \to (g_j)$ for the equivalence class of

$$\alpha : (f_i : X \to X_i)_I \to (g_j : Y \to Y_j)_J.$$

Observe that a pair $x_0, x_1 : \overline{X} \rightrightarrows X$ with the required weak universal property is clearly a pseudo-equivalence-relation and that if α is an arrow of $\mathbb{C}_{\text{reg}}$, there exists an arrow

$\overline{\alpha} : \overline{X} \to \overline{Y}$ such that $\alpha x_i = y_i\overline{\alpha}$, $i = 0, 1$. Also observe that if $\mathbb{C}$ is already the full subcategory of projectives of a regular category $\mathbb{E}$ in which every object can be embedded in a projective, then the previous definition of $\mathbb{C}_{\text{reg}}$ with families of arrows, and the one we gave at the begining with a single arrow, are in fact equivalent. We will see in the next theorem that in the converse direction, the need to take finite families of arrows as objects is due to the fact that we can then show that $\mathbb{C}_{\text{reg}}$ has finite products.

Consider now the Yoneda embedding in the presheaf category $\mathscr{P}\mathbb{C}$

$$y : \mathbb{C} \to \mathscr{P}\mathbb{C}$$

(ignoring size conditions, which for our finitary arguments are irrelevant), and observe that if $\mathscr{L} : \mathscr{D} \to \mathbb{C}$ is finite diagram in $\mathbb{C}$ and L is any weak limit of $\mathscr{L}$, then the canonical comparison $y(L) \to \lim y\mathscr{L}$ is a (regular) epi (that is, using the terminology which will be introduced in Section 3.1, the Yoneda embedding is a left covering functor). This simple observation is the key to show that $\mathbb{C}_{\text{reg}}$ can be described (up to equivalences) also as follows: take the full subcategory of the presheaf category $\mathscr{P}\mathbb{C}$ determined by those presheafs which appear at the same time as quotients of representables and as subobjects of finite (possibly empty) products of representables. In the proof of next theorem we will freely use both descriptions of $\mathbb{C}_{\text{reg}}$.

Theorem 8. $\mathbb{C}_{\text{reg}}$ *is a regular category.*

Proof. $\mathbb{C}_{\text{reg}}$ *is a left exact category*: given two objects (f_i) and (g_j) in $\mathbb{C}_{\text{reg}}$, their product is given by the following diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\pi_X} & X \times_w Y & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle f_i}\downarrow & & & & \downarrow{\scriptstyle g_j} \\ X_i & & & & Y_j \end{array}$$

where $X \xleftarrow{\pi_X} X \times_w Y \xrightarrow{\pi_Y} Y$ is a weak product in $\mathbb{C}$.

If T is a weak terminal object in $\mathbb{C}$, then the empty family of arrows with domain $T, (T \to)_\emptyset$ is the terminal object of $\mathbb{C}_{\text{reg}}$.

Consider now two parallel arrows $[\alpha], [\beta] : (f_i) \rightrightarrows (g_j)$ in $\mathbb{C}_{\text{reg}}$; their equalizer is given by the following diagram:

$$\begin{array}{ccccc} E & \xrightarrow{e} & X & \underset{\beta}{\overset{\alpha}{\rightrightarrows}} & Y \\ {\scriptstyle f_i e}\downarrow & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle g_j} \\ X_i & & X_i & & Y_j \end{array}$$

where $e : E \to X$ is a weak limit of the diagram

$$X \underset{g_j\beta}{\overset{g_j\alpha}{\rightrightarrows}} Y_j.$$

$\mathbb{C}_{reg}$ *has (regular epi)-(mono) factorization and regular epis are stable under pull-backs*: the previous description of finite limits in $\mathbb{C}_{reg}$ easily implies that the full inclusion of $\mathbb{C}_{reg}$ in the presheaf category on $\mathbb{C}$ is left exact. Moreover, $\mathbb{C}_{reg}$ is closed under subobjects in the presheaf category on $\mathbb{C}$. These two facts imply immediately that $\mathbb{C}_{reg}$ is a regular category (and that the inclusion in the presheaf category on $\mathbb{C}$ is an exact functor). □

Let $\mathbb{C}$ be a weakly lex category and $\mathbb{C}_{reg}$ its regular completion; the assignement

$$f : X \to Y \quad \rightsquigarrow \quad [f] : (1_X : X \to X) \to (1_Y : Y \to Y)$$

defines a functor $\Gamma : \mathbb{C} \to \mathbb{C}_{reg}$, which is *full and faithful and preserves monomorphic families.* It is now easy to show the following proposition.

Proposition 9. *Let $\Gamma : \mathbb{C} \to \mathbb{C}_{reg}$ be as previously defined. Then:*

(i) *$\mathbb{C}_{reg}$ has enough projectives and $\Gamma(\mathbb{C})$ is a projective cover of $\mathbb{C}_{reg}$;*

(ii) *each object of $\mathbb{C}_{reg}$ can be embedded in a product of projective objects;*

(iii) *a regular category $\mathbb{E}$ is the regular completion of a weakly lex category, if and only if it has a projective cover $\mathbb{P}$ such that every object can be embedded in a finite product of objects of $\mathbb{P}$. When this is the case, $\mathbb{E}$ is equivalent to $\mathbb{P}_{reg}$.*

Proof. The first and the second points follow from the exactness of the full inclusion of $\mathbb{C}_{reg}$ in the presheaf category on $\mathbb{C}$. The third point follows from the discussion at the beginning of this section. □

2.3. The exact completion of a regular category

The notion of a regular category is precisely the one that allows to develop the calculus of relations as an equational calculus over graphs. Defining a *relation R* from X to Y as a subobject $R \hookrightarrow X \times Y$, the existence in a regular category of regular images allows to define the *composite* of two relations as follows: if $S \hookrightarrow Y \times Z$ is another composable relation, then the composite SR is

$$SR = Im_{\pi_{X\times Z}}[\pi^*_{X\times Y}(R) \cap \pi^*_{Y\times Z}(S)],$$

where the π's denote projections from $X \times Y \times Z$ and the π^* denote the inverse image operators. Condition (3) of the definition of a regular category precisely means that the above composition is *associative*, determining in this way a *category* Rel($\mathbb{E}$) *of relations of* $\mathbb{E}$, whose identity morphisms are given by diagonal subobjects. Notice that Rel($\mathbb{E}$) has extra structure:

(i) a *local order* preserved by composition (in other words, Rel($\mathbb{E}$) is a locally ordered bicategory), which has finite intersections;

(ii) an *involution* $(\ )^{\circ}$, which is the identity on objects, and which preserves the local order;

(iii) an *embedding* $\mathbb{E} \to \mathrm{Rel}(\mathbb{E})$, given by the construction of the graph; we will freely confuse arrows of $\mathbb{E}$ with their graph in $\mathrm{Rel}(\mathbb{E})$. Working in $\mathrm{Rel}(\mathbb{E})$, we will denote (graphs of) arrows in $\mathbb{E}$ by lowercase latin letters and we will call them "*maps*".

The nice thing about this structure is that it allows to give purely algebraic proofs about facts in $\mathbb{E}$, by using the following lemma, whose proof is an exercise.

Lemma 10. *Let* $\mathbb{E}$ *be a regular category; then:*

(i) *an arrow* $R : X \to Y$ *of* $\mathrm{Rel}(\mathbb{E})$ *is the graph of an arrow of* $\mathbb{E}$ (*i.e. is a map*) *iff it has a right adjoint in the bicategory* $\mathrm{Rel}(\mathbb{E})$, *iff* R° *is a right adjoint, which simply means*

$$RR^{\circ} \le 1, \qquad R^{\circ}R \ge 1;$$

(ii) *an arrow* $f : X \to Y$ *of* $\mathbb{E}$ *is a mono iff* $f^{\circ}f = 1$ *and is a regular epi iff* $ff^{\circ} = 1$;

(iii) *for every relation* $R : X \to Y$ *there exist a pair of maps* f *and* g *such that*

$$R = gf^{\circ}, \qquad f^{\circ}f \cap g^{\circ}g = 1.$$

Such a pair is essentially unique ("*tabulation*" *of* R);

(iv) *a square in* $\mathbb{E}$

$$\begin{array}{ccc} X & \xrightarrow{k} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[g]{} & V \end{array}$$

is commutative iff $kh^{\circ} \le f^{\circ}g$ *and is a pullback iff* h, k *tabulate* $f^{\circ}g$; *in particular, the kernel pair of a map* $f : X \to Y$ *is a tabulation of the relation* $f^{\circ}f$;

(v) *the category* $\mathbb{E}$ *is exact if and only if in* $\mathrm{Rel}(\mathbb{E})$ *equivalence relations* (*i.e. endomorphisms* E *such that* $1 \le E$, $E \le E^{\circ}$, $EE \le E$), *considered as idempotents, split: there exist relations* P *and* Q *such that* $PQ = 1$ *and* $E = QP$ (*then, necessarily* P *is a map* p, *since* $1 \le E$, *and* $Q = p^{\circ}$).

An obvious question is to give a characterization of those locally ordered bicategories **B** which appear as $\mathrm{Rel}(\mathbb{E})$, $\mathbb{E}$ being necessarily the subcategory Map(**B**) of **B** determined by the arrows with right adjoint. Several answers have been given to this question, notably the one due to Freyd (see [11]), who has been able to give a characterization in terms of the following axiom:

$$RS \cap T \le R(S \cap R^{\circ}T)$$

(which he calls "*modular law*"), beside the other obvious axioms. A further analysis of the modular law has been carried out in [9], where the whole theory of relations has been reformulated in more flexible terms to cover other classes of examples (order ideals, abelian categories).

One of the uses of the theory of relations is to describe the left biadjoint to the forgetful 2-functor from the 2-category of exact categories to the one of regular categories, which we recall here, and which to our knowledge first appeared in [17]. The reader may consult [11] or [9] for further details. The point (v) of the previous lemma tells that a regular category is exact if and only if equivalence relations split in the category of relations. This suggests that, if $\mathbb{E}$ is only regular, we should define the exact completion by splitting the class of idempotents in the bicategory of relations, given by equivalence relations. This process will give us the bicategory of relations of the exact completion, and this last is then determined as its subcategory of maps. By using the characterization theorem of bicategories of relations of exact categories, one can prove that in fact we get in this way the "*exact completion*" $\mathbb{E}_{ex/reg}$ *of the regular category* $\mathbb{E}$. Explicitly:

Definition 11. Let $\mathbb{E}$ be a regular category; the exact completion of $\mathbb{E}$ can be described as follows:
- *objects*: An object of $\mathbb{E}_{ex/reg}$ is an equivalence relation $E \hookrightarrow X \times X$ in $\mathbb{E}$;
- *arrows*: An arrow $R : (X,E) \to (Y,F)$ is a relation $R : X \to Y$ in $\mathbb{E}$ such that

$$RE = R = FR$$

and

$$E \leq R^{\circ}R, \qquad RR^{\circ} \leq F;$$

- *composition*: Is the relation composition;
- *identities*: The identity on an object (X,E) is the equivalence relation E itself.

The fact that the previous definitions gives the exact completion of the regular category $\mathbb{E}$, precisely means the following. First observe that given a regular category $\mathbb{E}$, there exists a canonical embedding

$$\mathbb{E} \to \mathbb{E}_{ex/reg}$$

sending each object X in the discrete equivalence relation on it, and that it is an exact functor.

Proposition 12. *For each regular category* $\mathbb{E}$ *and each exact category* $\mathbb{A}$, *the embedding*

$$\mathbb{E} \to \mathbb{E}_{ex/reg}$$

induces an equivalence between the category of exact functors from $\mathbb{E}$ *to* $\mathbb{A}$ *and the category of exact functors from* $\mathbb{E}_{ex/reg}$ *to* $\mathbb{A}$.

Let us finish by recalling that the embedding of the 2-category of exact categories into the one of regular categories is *full*, and hence that the exact completion of a regular category is an *idempotent* process.

2.4. The exact completion of a weakly lex category

Starting with a weakly lex category $\mathbb{C}$, we take the regular completion $\mathbb{E} = \mathbb{C}_{reg}$ described in Section 2.2 and we can apply to it the exact completion $\mathbb{E}_{ex/reg}$ described in the previous section, so obtaining an exact category

$$\mathbb{C}_{ex} = (\mathbb{C}_{reg})_{ex/reg}.$$

The exact category so constructed is the exact completion of the weakly lex category $\mathbb{C}$. The following lemma provides an explicit description of $\mathbb{C}_{ex}$.

Lemma 13. *When $\mathbb{E}$ is a regular category, the canonical embedding $\mathbb{E} \to \mathbb{E}_{ex/reg}$ takes projective covers of $\mathbb{E}$ into projective covers of $\mathbb{E}_{ex/reg}$.*

Proof. Just observe that a regular epi $P : (X,E) \to (Y,\Delta_Y)$ in $\mathbb{E}_{ex/reg}$ whose codomain is a discrete equivalence relation is a regular epi in $\mathbb{E}$; hence, when the codomain is projective in $\mathbb{E}$, then it has a section s in $\mathbb{E}$, and the relation Es is a section of P in $\mathbb{E}_{ex/reg}$. The remark that every object (X,E) of $\mathbb{E}_{ex/reg}$ is covered by an object of $\mathbb{E}$, namely by (X,Δ_X), concludes the proof. □

From this lemma and Proposition 9, it follows that when $\mathbb{E}$ is the regular completion $\mathbb{E} = \mathbb{C}_{reg}$, then $\mathbb{C}$ is a projective cover of $\mathbb{C}_{ex}$. Hence, from the discussion in Section 2.1, we have that $\mathbb{C}_{ex}$ can be described as follows.

Definition 14. Let $\mathbb{C}$ be a weakly lex category; the category $\mathbb{C}_{ex}$ (the "*exact completion* of $\mathbb{C}$") has

- *objects*: An object of $\mathbb{C}_{ex}$ is a pseudo-equivalence relation in $\mathbb{C}$

$$r_0, r_1 : R \rightrightarrows X;$$

- *arrows*: An arrow between two objects

$$r_0, r_1 : R \rightrightarrows X \quad \text{and} \quad s_0, s_1 : S \rightrightarrows Y$$

of $\mathbb{C}_{ex}$ is an equivalence class of pairs of compatible arrows $(f,\bar{f})$ as in the following diagram:

$$\begin{array}{ccc} R & \xrightarrow{\bar{f}} & S \\ {\scriptstyle r_0}\downarrow\downarrow{\scriptstyle r_1} & & {\scriptstyle s_0}\downarrow\downarrow{\scriptstyle s_1} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where the pair $(f,\bar{f})$ is said to be compatible if $s_0\bar{f} = f r_0$ and $s_1\bar{f} = f r_1$; such two pairs $(f,\bar{f})$ and $(g,\bar{g})$ are considered to be equivalent if there exists an arrow (an "half-homotopy") $\Sigma : X \to S$ such that $s_0\,\Sigma = f$ and $s_1\,\Sigma = g$;
- *composition and identities*: The obvious ones.

Observe that the "half-homotopy" relation is an equivalence relation precisely because the objects are pseudo-equivalence relations. The equivalence class of $(f,\bar{f})$ will be denoted simply by $[f]$, since by the half-homotopy relation, each two pairs with the same first component are in fact equivalent. Observe also that no size conditions are requested on $\mathbb{C}$ to construct $\mathbb{C}_{\mathrm{ex}}$ and that $\mathbb{C}_{\mathrm{ex}}$ is (locally) small if $\mathbb{C}$ is (locally) small.

As we observed in Section 2.2 for the regular completion, the exact completion of $\mathbb{C}$ also admits an equivalent description as a full subcategory of the presheaf category on $\mathbb{C}$: $\mathbb{C}_{\mathrm{ex}}$ is in fact equivalent to the full subcategory of the presheaf category on $\mathbb{C}$ determined by those quotients of representables whose kernel can be covered by a single representable.

The following proposition, whose proof is straightforward, sumarizes the properties of $\mathbb{C}_{\mathrm{ex}}$ we know so far.

Proposition 15. *Let $\mathbb{C}$ be a weakly lex category and $\mathbb{C}_{\mathrm{ex}}$ its exact completion.*

(i) *The functor*

$$\Gamma : \mathbb{C} \to \mathbb{C}_{\mathrm{ex}}$$

sending each object into the pair of identities, is full and faithful and preserves monomorphic families. Moreover, for each object Y of $\mathbb{C}$, ΓY is a projective object in $\mathbb{C}_{\mathrm{ex}}$.

(ii) *The image $\Gamma(\mathbb{C})$ generates $\mathbb{C}_{\mathrm{ex}}$ via coequalizers, that is, if*

$$[f,\bar{f}] : (r_0, r_1 : R \rightrightarrows X) \to (s_0, s_1 : S \rightrightarrows Y)$$

is an arrow in $\mathbb{C}_{\mathrm{ex}}$, then in the following diagram in $\mathbb{C}_{\mathrm{ex}}$ the two horizontal lines are coequalizers and the last vertical arrow is the unique extension to the quotient

$$\begin{array}{ccccc}
\Gamma R & \underset{\Gamma r_1}{\overset{\Gamma r_0}{\rightrightarrows}} & \Gamma X & \xrightarrow{[1_X, r_R]} & (R \rightrightarrows X) \\
{\scriptstyle \Gamma\bar{f}}\downarrow & & \downarrow{\scriptstyle \Gamma f} & & \downarrow{\scriptstyle [f]} \\
\Gamma S & \underset{\Gamma s_1}{\overset{\Gamma s_0}{\rightrightarrows}} & \Gamma Y & \xrightarrow[{[1_Y, r_S]}]{} & (S \rightrightarrows Y).
\end{array}$$

(iii) *The image $\Gamma(\mathbb{C})$ of the functor $\Gamma : \mathbb{C} \to \mathbb{C}_{\mathrm{ex}}$ is a projective cover of $\mathbb{C}_{\mathrm{ex}}$, so that $\mathbb{C}_{\mathrm{ex}}$ has enough projectives.*

Let us observe that there is no reason why $\Gamma : \mathbb{C} \to \mathbb{C}_{\mathrm{ex}}$ should send weak limits into weak limits; for example, if T is a weak terminal in $\mathbb{C}$, the terminal in $\mathbb{C}_{\mathrm{ex}}$ is given by the two projections $T \times_w T \rightrightarrows T$ and there is no reason to have an arrow in $\mathbb{C}_{\mathrm{ex}}$

from $T \times_w T \rightrightarrows T$ to $\Gamma(T)$. An explicit counterexample will be given in Section 3.3. Nevertheless, let us observe that the corestriction of $\Gamma : \mathbb{C} \to \mathbb{C}_{ex}$ to the full subcategory of projective objects of $\mathbb{C}_{ex}$ preserves finite weak limits. This can be easily deduced from the fact that an object $r_0, r_1 : R \rightrightarrows X$ is projective if and only if it is *contractible*, that is there exists an arrow $\Sigma : X \to R$ such that $r_0 \Sigma = 1_X$ and $r_1 \Sigma r_0 = r_1 \Sigma r_1$. Moreover, Γ preserves all the honest finite limits which turn out to exist in $\mathbb{C}$; this is an easy consequence of a more general result contained in the next section.

We can sumarize the whole discussion on the exact completion in the following theorem, which clarifies the nature of the property of an exact category of having "*enough projectives*".

Theorem 16. *Exact categories with enough projectives are the exact completions of the weakly lex categories of their projectives (more generally, of any of their projective covers) and, conversely, each weakly lex category in which idempotents split appears as the full subcategory of the projectives of an exact category with enough projectives, namely of its exact completion.*

We end this section with some remarks on the exact completion of a regular category. The following proposition is quite straightforward.

Proposition 17. *Let $\mathbb{E}$ be a regular category; $\mathbb{E}$ is a reflective subcategory of the exact completion $\mathbb{E}_{ex/reg}$ if and only if $\mathbb{E}$ has coequalizers of equivalence relations.*

In particular, when the regular category $\mathbb{E}$ is the regular completion of a weakly lex category $\mathbb{C}$, the condition on $\mathbb{E}$ of having coequalizers of equivalence relation can be completely expressed in terms of $\mathbb{C}$ as follows.

Proposition 18. *Let $\mathbb{C}$ be a weakly lex category; its regular completion $\mathbb{C}_{reg}$ has coequalizers of equivalence relations if and only if the following condition holds: given a pseudo equivalence-relation $r_0, r_1 : R \rightrightarrows X$ in $\mathbb{C}$, there exists a finite family $(q_i : X \to Q_i)_I$ of arrows in $\mathbb{C}$ such that*

(1) $q_i r_0 = q_i r_1$, for all $i \in I$;

(2) if $f : X \to Y$ is such that $f r_0 = f r_1$, then $f q_0 = f q_1$ (where

$$q_0,\ q_1 : \overline{X} \rightrightarrows X$$

is weak universal such that $q_i q_0 = q_i q_1$, for all $i \in I$).

3. The universality of the completions

3.1. Left covering functors

Looking at the characterization of exact categories with enough projectives of the previous section, one would be tempted to believe that there is a biequivalence between the 2-category of exact categories with enough projectives and exact functors between them, and the 2-category of weakly lex categories in which idempotents split and weakly lex functors between them, i.e. functors which send weak limits into weak limits. Unfortunately, the situation is not so simple, the precise nature of the universal property of the exact completion construction being not quite the expected one. Keeping in mind the proof of Proposition 4, the main definition turns out to be the following.

Definition 19. Consider a functor $F : \mathbb{C} \to \mathbb{A}$ with $\mathbb{C}$ weakly lex and $\mathbb{A}$ a regular category; we say that F is "left covering" if, for all functors $\mathscr{L} : \mathscr{D} \to \mathbb{C}$ defined on a finite category $\mathscr{D}$ and for all weak limits

$$\operatorname{wlim} \mathscr{L} = (\pi_D : L \to \mathscr{L}D)_{D \in \mathscr{D}_0},$$

the canonical factorization $p : FL \to \tilde{L}$ is a regular epimorphism. Here p is the unique arrow such that $F\pi_D = \tilde{\pi}_D p$, where

$$(\tilde{\pi}_D : \tilde{L} \to F(\mathscr{L}D))_{D \in \mathscr{D}_0} = \lim F\mathscr{L}.$$

Observe that in the previous definition the second "for all" can be equivalently replaced by a "for one". In fact, if

$$(\pi_D : L \to \mathscr{L}D)_{D \in \mathscr{D}_0} \quad \text{and} \quad (\pi'_D : L' \to \mathscr{L}D)_{D \in \mathscr{D}_0}$$

are weak limits of $\mathscr{L} : \mathscr{D} \to \mathbb{C}$, then there exists a factorization $t : L \to L'$; now, if the factorization $p : FL \to \tilde{L}$ is a regular epimorphism, even the factorization $p' : FL' \to \tilde{L}$ must be a regular epimorphism because $p'Ft = p$.

The previous definition can be adapted to a functor $F : \mathbb{C} \to \mathbb{A}$ with $\mathbb{A}$ only lex, provided that we require that the map p is a *strong epi*. However, even assuming that p is a strong epi, if we assume that $\mathbb{A}$ is only weakly lex, this definition is not stable under composition (look once again at the example in Section 3.3).

Proposition 20. *Consider a functor $F : \mathbb{C} \to \mathbb{A}$ with $\mathbb{C}$ weakly lex and $\mathbb{A}$ a lex category; consider also the following conditions:*
(1) F is left covering;
(2) F is weakly lex;
(3) F is left exact.
One has that (2) *implies* (1); *moreover, if $\mathbb{C}$ is left exact, then the three conditions are equivalent.*

Proof. To prove that (2) implies (1) and, if $\mathbb{C}$ is left exact, that (3) implies (2), it suffices to use the fact that, if there exists the honest limit, then weak limits are exactly the coretracts of the honest one. Now, assume $\mathbb{C}$ left exact and F left covering; let us start showing that F preserves the terminal object T of $\mathbb{C}$: by hypothesis, the unique arrow $q : FT \to \tilde{T}$ (where $\tilde{T}$ is the terminal object of $\mathbb{A}$) is a strong epimorphism; in $\mathbb{C}$, one has that $T \xleftarrow{1_T} T \xrightarrow{1_T} T$ is the product of T with itself, so that the unique factorization $FT \to FT \times FT$ is a (strong) epimorphism and then the two projections $FT \xleftarrow{\pi_1} FT \times FT \xrightarrow{\pi_2} FT$ are equal. But the pair π_1, π_2 is the kernel pair of q, so that q is a mono and then an iso.

To show that F preserves not empty finite limits, we need a lemma, whose proof (up to some minor modifications) is based on the same argument used in 1.829 of [11].

Lemma 21. *Let $F : \mathbb{C} \to \mathbb{A}$ be a left covering functor; F preserves the finite monomorphic families.*

Let us come back to the proof of Proposition 20: $F : \mathbb{C} \to \mathbb{A}$ is a left covering functor between left exact categories and we have to prove that F preserves non-empty finite limits. Consider a functor $\mathscr{L} : \mathscr{D} \to \mathbb{C}$ defined on a non-empty finite category $\mathscr{D}$; consider also $\lim \mathscr{L} = (\pi_D : L \to \mathscr{L}D)_{D \in \mathscr{D}_0}$ and $\lim F\mathscr{L} = (\tilde{\pi}_D : \tilde{L} \to F(\mathscr{L}D))_{D \in \mathscr{D}_0}$: the family $(\pi_D : L \to \mathscr{L}D)_{D \in \mathscr{D}_0}$ is monomorphic so that also the family $(F\pi_D : FL \to F(\mathscr{L}D))_{D \in \mathscr{D}_0}$ is monomorphic by Lemma 21; for all $D \in \mathscr{D}_0$, we have that $F\pi_D = \tilde{\pi}_D\, p$, so that the unique factorization $p : FL \to \tilde{L}$ is a monomorphism; but, by hypothesis, it is a strong epimorphism, so it is an isomorphism. □

Let us remark that in the proof of Proposition 20 we have established a more general fact: a left covering functor from a weakly lex category to a lex category preserves all the finite limits which turn out to exist in the domain (i.e. being left covering is a kind of *flatness*).

In the next proposition and corollary we complete the comparison between left covering functors and weakly lex functors. The proofs are simple and will be omitted. Let us only make clear that we call F "*flat*" if, for each functor $\mathscr{L} : \mathscr{D} \to \mathbb{C}$ with $\mathscr{D}$ finite, and for each cone $(\tilde{\pi}_D : \tilde{L} \to F(\mathscr{L}D))_{D \in \mathscr{D}_0}$ over $F\mathscr{L}$ in $\mathbb{A}$, there exist a cone $(\pi_D : L \to \mathscr{L}D)_{D \in \mathscr{D}_0}$ over $\mathscr{L}$ in $\mathbb{C}$ and a factorization $t : \tilde{L} \to F(L)$. Observe that this condition does not require the existence of weak limits in $\mathbb{C}$. It means that, for each $A \in \mathbb{A}$, the comma category (A, F) is cofiltered.

Proposition 22. *Consider a functor $F : \mathbb{C} \to \mathbb{A}$ with $\mathbb{C}$ weakly lex and $\mathbb{A}$ lex. Suppose that F factors through the full subcategory $P(\mathbb{A})$ of projective objects (with respect to strong epimorphisms) of $\mathbb{A}$ and call $F' : \mathbb{C} \to P(\mathbb{A})$ its corestriction:*

(1) *if F is left covering, then F' preserves finite weak limits of $\mathbb{C}$;*

(2) *if $\mathbb{A}$ has enough projectives and F' is weakly lex, then F is left covering.*

Corollary 23. *Let $F : \mathbb{C} \to \mathbb{A}$ be a functor with $\mathbb{C}$ weakly lex and $\mathbb{A}$ lex; if the axiom of choice holds in $\mathbb{A}$ (that is each object is projective), then F is weakly lex* *if and only if it is left covering, if and only if it is flat. If $\mathbb{A}$ is the category of sets, then F is left covering if and only if it is a filtered colimit of representable functors.*

In the next two propositions, whose proofs are also simple, we look at the stability of the notion of left covering functor.

Proposition 24. *Let $\mathbb{B} \xrightarrow{G} \mathbb{C} \xrightarrow{F} \mathbb{A}$ be two functors with $\mathbb{B}$ and $\mathbb{C}$ weakly lex and $\mathbb{A}$ regular; if G is weakly lex and F is left covering, then the composition $FG : \mathbb{B} \to \mathbb{A}$ is left covering.*

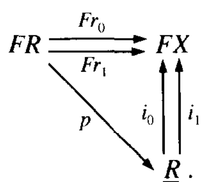
Proposition 25. *Let $\mathbb{C} \xrightarrow{F} \mathbb{A} \xrightarrow{G} \mathbb{B}$ be two functors with $\mathbb{C}$ weakly lex and $\mathbb{A}$ and $\mathbb{B}$ regular; if F is left covering and G is exact, then the composition $GF : \mathbb{C} \to \mathbb{B}$ is left covering.*

The next theorem gives the most relevant property of left covering functors:

Theorem 26. *Consider a left covering functor $F : \mathbb{C} \to \mathbb{A}$, and let*

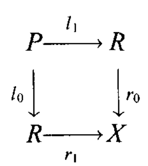
$$r_0, r_1 : R \rightrightarrows X$$

be a pseudo-equivalence relation in $\mathbb{C}$; consider the (regular epi)-(jointly monic) factorization of its image by F

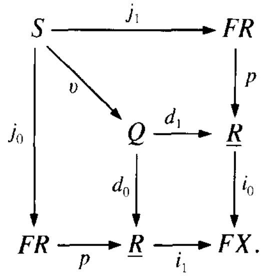


Then $i_0, i_1 : \underline{R} \rightrightarrows FX$ is an equivalence relation in $\mathbb{A}$.

Proof. The reflexivity and the symmetry of $i_0, i_1 : \underline{R} \rightrightarrows FX$ are easy. For the transitivity, consider a weak pullback



and the transitivity morphism $t_R : P \to R$ (that is $r_0 t_R = r_0 l_0$ and $r_1 t_R = r_1 l_1$). Consider now the following diagram in $\mathbb{A}$, in which both squares are pullbacks so that the factorization v is a regular epi



Using now the fact that the functor $F : \mathbb{C} \to \mathbb{A}$ is left covering, we have that the factorization $q : FP \to S$ such that $j_0 q = Fl_0$ and $j_1 q = Fl_1$ is a regular epi. A diagram chasing shows now that

$$\langle i_0 d_0, i_1 d_1 \rangle v_q = \langle i_0, i_1 \rangle pFt_R.$$

But vq is a regular epi and $\langle i_0, i_1 \rangle$ is a mono, so that there exists an arrow $h : Q \to \underline{R}$ such that $\langle i_0 d_0, i_1 d_1 \rangle = \langle i_0, i_1 \rangle h$. This shows that the relation $i_0, i_1 : \underline{R} \rightrightarrows FX$ is transitive. □

Observe that this theorem applies when as $F : \mathbb{C} \to \mathbb{A}$ we consider the full inclusion of a projective cover of a regular category and, in particular, the embedding $\Gamma : \mathbb{C} \to \mathbb{C}_{\text{ex}}$.

The next proposition is crucial to make handy the notion of left covering functor.

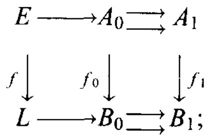
Proposition 27. *Consider a functor $F : \mathbb{C} \to \mathbb{A}$ with $\mathbb{C}$ weakly lex and $\mathbb{A}$ regular; if F is left covering with respect to binary products, equalizers of pairs of parallel arrows and terminal object, then it is left covering.*

Proof. We first need a lemma on regular categories, whose proof is a simple exercise.

Lemma 28. *Let $\mathbb{A}$ be a regular category:*

(1) *if f_0 and f_1 are regular epis, then $f_0 \times f_1$ is a regular epi;*

(2) *in the following commutative diagram, where the two horizontal lines are equalizers, if f_0 is a regular epi and f_1 a mono, then the unique factorization f is a regular epi:*



(3) *if the following diagram is commutative, f_0 and f_1 are regular epis and f is a mono, then the unique factorization from the pullback of a_0 and a_1 to the pullback of b_0 and b_1 is a regular epi*:

$$\begin{array}{ccccc} A_0 & \xrightarrow{a_0} & A & \xleftarrow{a_1} & A_1 \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ B_0 & \xrightarrow[b_0]{} & B & \xleftarrow[b_1]{} & B_1. \end{array}$$

Coming back to Proposition 27, using the previous lemma it is straightforward to show that F is left covering with respect to n-ary products, equalizers and pullbacks. Consider now a functor $\mathscr{L} : \mathscr{D} \to \mathbb{C}$ defined on a finite category $\mathscr{D}$. The result follows from the previous lemma, using the description of the weak limit of $\mathscr{L}$ given in the proof of Proposition 1, and computing in the same way the limit of $F\mathscr{L}$. □

3.2. *The universality of the completions*

In this section we show that the embeddings $\Gamma : \mathbb{C} \to \mathbb{C}_{\mathrm{reg}}$ and $\Gamma : \mathbb{C} \to \mathbb{C}_{\mathrm{ex}}$ of a weakly lex category in its regular and exact completions are universal in the sense of the following theorem.

Theorem 29. *Let $\mathbb{C}$ be a weakly lex category and $\mathbb{A}$ be a regular one, and let $\Gamma : \mathbb{C} \to \mathbb{C}_{\mathrm{reg}}$ be the regular completion of $\mathbb{C}$; then Γ induces an equivalence between the category of left covering functors from $\mathbb{C}$ to $\mathbb{A}$, and the category of exact functors from $\mathbb{C}_{\mathrm{reg}}$ to $\mathbb{A}$. The same holds for the exact completion, with respect to any exact category $\mathbb{A}$.*

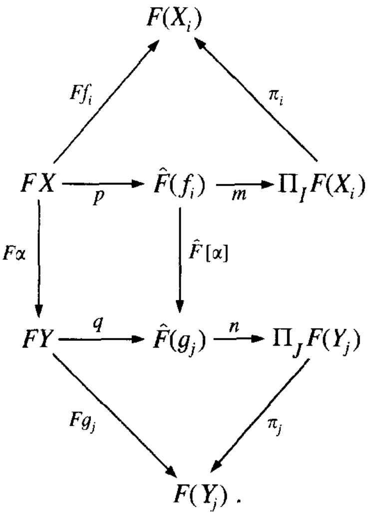
Proof. Let us start by showing that, given a left covering functor $F : \mathbb{C} \to \mathbb{A}$, where $\mathbb{A}$ is a regular category, there exists an essentially unique exact extension $\hat{F} : \mathbb{C}_{\mathrm{reg}} \to \mathbb{A}$. Consider an arrow in $\mathbb{C}_{\mathrm{reg}}$

$$[\alpha] : (f_i) \to (g_j);$$

keeping in mind the description of $\mathbb{C}_{\mathrm{reg}}$ as a full subcategory of the presheaf category on $\mathbb{C}$ given in Section 2.2, if $\hat{F} : \mathbb{C}_{\mathrm{reg}} \to \mathbb{A}$ preserves regular epis, mono's and finite products and if $\hat{F}\Gamma \simeq F$, then $\hat{F}(f_i)$ must be the image in $\mathbb{A}$ of

$$\langle Ff_i \rangle : FX \to \prod_I F(X_i)$$

and $\hat{F}[\alpha]$ must be the unique extension to the images as in the following diagram:



This gives the essential uniqueness of $\hat{F}$.

Existence of $\hat{F}$: The existence of $\hat{F}$ on the objects depends only on the regularity of $\mathbb{A}$; if the extension $\hat{F}[\alpha] : \hat{F}(f_i) \to \hat{F}(g_j)$ exists, then the functoriality of $\hat{F}$ and the fact that $\hat{F}\Gamma \simeq F$ are obvious. As for the arrows,

$$[\alpha] : (f_i) \to (g_j),$$

by the definition of an arrow in $\mathbb{C}_{\mathrm{reg}}$, we have that $g_j\,\alpha x_0 = g_j\,\alpha x_1$, for all $j \in J$, x_0, x_1 being a joint weak kernel pair of (f_i). Now take the kernel pair $p_0, p_1 : N(p) \rightrightarrows FX$ of $\langle Ff_i \rangle : FX \to \prod_I F(X_i)$ (that is of p); as $f_i x_0 = f_i x_1$, for all $i \in I$, and hence $\langle Ff_i \rangle Fx_0 = \langle Ff_i \rangle Fx_1$, there exists a factorization $x : F\overline{X} \to N(p)$ such that $p_0 x = Fx_0$ and $p_1 x = Fx_1$; The point is that the map x is a regular epi, since F is a *left covering functor*. We are now in the following situation:

$$\begin{array}{ccccccc} F\overline{X} & \xrightarrow{x} & N(p) & \overset{p_0}{\underset{p_1}{\rightrightarrows}} & FX & \xrightarrow{p} & \hat{F}(f_i) \\ & & & & \big\downarrow{\scriptstyle F\alpha} & & \\ & & & & FY & \xrightarrow[q]{} & \hat{F}(g_j). \end{array}$$

Being

$$N(p) \overset{p_0}{\underset{p_1}{\rightrightarrows}} FX \xrightarrow{p} \hat{F}(f_i)$$

a coequalizer diagram, to obtain the arrow $\hat{F}[\alpha] : \hat{F}(f_i) \to \hat{F}(g_j)$ it suffices to show that $q\,F\alpha\,p_0 = q\,F\alpha\,p_1$. Let us show that this is the case when we compose with x on the right and with n on the left; this is equivalent to show that $Fg_j\,F\alpha\,Fx_0 = Fg_j\,F\alpha\,Fx_1$, for all $j \in J$, and this follows from the condition on α to be an arrow in $\mathbb{C}_{\mathrm{reg}}$. Observe that this also means that

$$F\overline{X} \underset{Fx_1}{\overset{Fx_0}{\rightrightarrows}} FX \xrightarrow{\;p\;} \hat{F}(f_i)$$

is a coequalizer, because x is an epimorphism.

$\hat{F} : \mathbb{C}_{\mathrm{reg}} \to \mathbb{A}$ *is exact*: It is easy to show that $\hat{F}$ preserves regular epis, using the fact that regular epis in $\mathbb{C}_{\mathrm{reg}}$ are of the form

$$[1_X] : (f_i : X \to X_i) \to (g_j : X \to Y_j).$$

To show that $\hat{F}$ preserves finite limits, it is enough to show that $\hat{F}$ is left covering with respect to binary products, equalizers and terminal objects (see Propositions 20 and 27). As for products, consider two objects (f_i) and (g_j) in $\mathbb{C}_{\mathrm{reg}}$ and apply $\hat{F}$ to their product. By definition of $\hat{F}$, we obtain the following commutative diagram in $\mathbb{A}$:

$$\begin{array}{ccccc}
FX & \xleftarrow{F(\pi_X)} & F(X \times Y) & \xrightarrow{F(\pi_Y)} & FY \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} \\
\hat{F}(f_i) & \underset{\hat{F}[\pi_X]}{\longleftarrow} & \hat{F}((f_i) \times (g_j)) & \underset{\hat{F}[\pi_Y]}{\longrightarrow} & \hat{F}(g_j).
\end{array}$$

Consider again the canonical factorization $s : F(X \times Y) \to FX \times FY$ which is a regular epi (because F is left covering). Now the canonical factorization $t : \hat{F}((f_i) \times (g_j)) \to \hat{F}(f_i) \times \hat{F}(g_j)$ satisfies $tr = (p \times q)s$, and then it is a regular epi because s, p and q are regular epis.

Now, let $[\alpha], [\beta] : (f_i) \rightrightarrows (g_j)$ be two parallel arrows in $\mathbb{C}_{\mathrm{reg}}$ and let

$$\begin{array}{ccccc}
E & \xrightarrow{\;e\;} & X & \underset{\beta}{\overset{\alpha}{\rightrightarrows}} & Y \\
{\scriptstyle f_i e}\downarrow & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle g_j} \\
X_i & & X_i & & Y_j
\end{array}$$

be the coequalizer (with the notations of Theorem 8). By definition of $\hat{F}$, we obtain the following commutative diagram in $\mathbb{A}$:

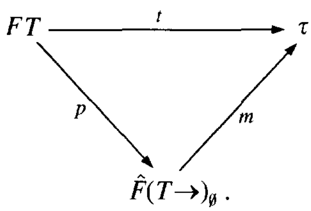
$$\begin{array}{ccccc}
FE & \xrightarrow{\;Fe\;} & FX & \underset{F\beta}{\overset{F\alpha}{\rightrightarrows}} & FY \\
{\scriptstyle r}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\
\hat{F}(f_i e) & \underset{\hat{F}[e]}{\longrightarrow} & \hat{F}(f_i) & \underset{\hat{F}[\beta]}{\overset{\hat{F}[\alpha]}{\rightrightarrows}} & \hat{F}(g_j).
\end{array}$$

Consider now the equalizer $h : H \to FX$ of $(qF\alpha, qF\beta)$ and the equalizer $l : L \to \hat{F}(f_i)$ of $(\hat{F}[\alpha], \hat{F}[\beta])$. The equalizer h is clearly also the limit of the diagram:

$$FX \underset{F(g_j\beta)}{\overset{F(g_j\alpha)}{\rightrightarrows}} FY_j$$

so that there exists a regular epi $t : FE \to H$ such that $ht = Fe$ (because F is left covering). Moreover, the uniqe arrow $\tau : H \to L$ such that $l\tau = ph$ is a regular epi (see Lemma 28). Now let $v : \hat{F}(f_i e) \to L$ be the unique arrow such that $lv = \hat{F}[e]$. One has that $vr = \tau t$ (just compose with l), so that v also is a regular epi.

It remains to prove that $\hat{F}$ is left covering with respect to the terminal object $(T \to)_\emptyset$ of $\mathbb{C}_{\text{reg}}$. It is clear, because $\hat{F}(T \to)_\emptyset$ is the image of the unique arrow $t : FT \to \tau$, where τ is the terminal object of $\mathbb{A}$



But, by assumption on F, t is a regular epi and hence also m is a regular epi.

The fact that composition with Γ is a full and faithful functor will follow from Proposition 31.

Finally, to obtain the universal property of the exact completion, it suffices to combine the universal property of the regular completion just shown, with the universal property of the exact completion of a regular category stated in Proposition 12. Using the description of the exact completion of a weakly lex category $\mathbb{C}$ given in Definition 14, one can show that the exact extension $\hat{F}$ of a left covering functor $F : \mathbb{C} \to \mathbb{A}$ from $\mathbb{C}$ to an exact category $\mathbb{A}$ is defined by sending each pseudo-equivalence relation in $\mathbb{C}$ into the coequalizer in $\mathbb{A}$ of its image under F, which is an equivalence relation in $\mathbb{A}$ by Theorem 26. □

Let us observe that the equivalences $\mathbb{P}_{\text{reg}} \simeq \mathbb{A}$ and $\mathbb{P}_{\text{ex}} \simeq \mathbb{A}$ described in Sections 2.2 and 2.1 are nothing but the exact extension of the full inclusion of the projective cover $\mathbb{P}$ in the regular (respectively exact) category $\mathbb{A}$.

From the fact that the embeddings $\Gamma : \mathbb{C} \to \mathbb{C}_{\text{reg}}$ and $\Gamma : \mathbb{C} \to \mathbb{C}_{\text{ex}}$ are themselves left covering functors (and using the stability stated in Proposition 24), one has immediately the following corollary

Corollary 30. *The regular and the exact completions are determined up to equivalences by the universal property of Theorem* 29.

3.3. Further remarks

As a left covering functor defined on a left exact category is exactly a left exact functor (Proposition 20), we have, as a particular case of Theorem 29, the main theorem contained in [7]; as it is shown there, the universal property of the exact completion of a left exact category becomes part of the left biadjoint to the obvious forgetful

2-functor

$$\mathrm{EX} \to \mathrm{LEX},$$

where EX is the 2-category of exact categories and exact functors and LEX is the 2-category of left exact categories and left exact functors.

The question naturally arising is then whether, with a good choice of morphisms between weakly lex categories, the universal property stated in Theorem 29 becomes part of the analogous biadjunction between exact categories and weakly lex ones.

The answer is *negative*. For, suppose that we have organized the weakly lex categories in a 2-category, say WLEX, and that the exact completion of a weakly lex category $\Gamma : \mathbb{C} \to \mathbb{C}_{ex}$ is the unit of a biadjunction

$$\mathrm{EX} \leftrightarrows \mathrm{WLEX}.$$

Then, if $\mathbb{A} \in \mathrm{EX}$ and $\mathbb{C} \in \mathrm{WLEX}$, the category of morphisms $\mathrm{WLEX}(\mathbb{C}, \mathbb{A})$ must be equivalent, via the composition with $\Gamma : \mathbb{C} \to \mathbb{C}_{ex}$, to the category of exact functors from $\mathbb{C}_{ex}$ to $\mathbb{A}$, which is equivalent to the category of left covering functors from $\mathbb{C}$ to $\mathbb{A}$. Of course, the unit $\Gamma : \mathbb{C} \to \mathbb{C}_{ex}$ is in $\mathrm{WLEX}(\mathbb{C}, \mathbb{C}_{ex})$; but now, if we perform again the exact completion, we get a functor

$$\mathbb{C} \xrightarrow{\Gamma} \mathbb{C}_{ex} \xrightarrow{\Gamma} (\mathbb{C}_{ex})_{ex}$$

which must be in $\mathrm{WLEX}(\mathbb{C}, (\mathbb{C}_{ex})_{ex})$ (being the composition of two morphisms). So that, $(\mathbb{C}_{ex})_{ex}$ being exact, the functor $\mathbb{C} \xrightarrow{\Gamma} \mathbb{C}_{ex} \xrightarrow{\Gamma} (\mathbb{C}_{ex})_{ex}$ must be a left covering functor, by Proposition 25. But, in general, this is not the case, as the following example shows. Using Theorem 16, as composition

$$\mathbb{C} \xrightarrow{\Gamma} \mathbb{C}_{ex} \xrightarrow{\Gamma} (\mathbb{C}_{ex})_{ex}$$

we can choose $\mathbb{P} \xrightarrow{i} \mathbb{A} \xrightarrow{\Gamma} (\mathbb{A})_{ex}$, where $\mathbb{A}$ is an exact category, $\mathbb{P}$ a projective cover of $\mathbb{A}$ and $i : \mathbb{P} \to \mathbb{A}$ the full inclusion. Now cover the terminal object τ of $\mathbb{A}$ with an object T of $\mathbb{P}$ and a regular epi $t : T \to \tau$, so that T is a weak terminal in $\mathbb{P}$. $\Gamma : \mathbb{A} \to \mathbb{A}_{ex}$ is left exact, so that $\Gamma\tau$ is a terminal object of $\mathbb{A}_{ex}$. If the composition $\Gamma i : \mathbb{P} \to \mathbb{A}_{ex}$ is left covering, then the unique arrow from $\Gamma(iT)$ to $\Gamma\tau$, that is $\Gamma t : \Gamma T \to \Gamma\tau$, is a regular epi. But $\Gamma\tau$ is projective in $\mathbb{A}_{ex}$ so that the regular epi Γt has a section, say $s : \Gamma\tau \to \Gamma T$, in $\mathbb{A}_{ex}$. Since $\Gamma : \mathbb{A} \to \mathbb{A}_{ex}$ is full and faithful, this implies that $t : T \to \tau$ has a section in $\mathbb{A}$.

To show that, in general, this is not true, choose $\mathbb{A}$ as the category of rings and $\mathbb{P}$ as the full subcategory of projective rings; as $t : T \to \tau$, one can choose the unique morphism $\mathbb{Z} \to (0 = 1)$ which, obviously, has no section in $\mathbb{A}$. The inclusion of the projective rings into the category of rings gives us also an example of functor $\Gamma : \mathbb{C} \to \mathbb{C}_{ex}$ which does not preserve the finite weak limits, as pointed out in Section 2.4.

Of course, there could be a different construction which provides a biadjoint, depending upon the choice of morphisms between weakly lex categories. We only know that our construction does not provide a biadjoint, whatever is the class of morphisms we choose between weakly lex categories.

We conclude this part with some elementary facts on the exact completion which will be useful for applications.

Proposition 31. *Let $\Gamma : \mathbb{C} \to \mathbb{C}_{\mathrm{reg}}$ be the regular completion of a weakly lex category $\mathbb{C}$ and consider a left covering functor $F: \mathbb{C} \to \mathbb{A}$ with $\mathbb{A}$ regular; then the exact extension $\hat{F} : \mathbb{C}_{\mathrm{reg}} \to \mathbb{A}$ described in Theorem 29 is a left Kan-extension of F along Γ. A similar result holds for the exact completion.*

Proof. Straightforward from Proposition 9 and from Proposition 15. □

From Theorem 16 we are able to recognize free exact categories; the analogous result for functors is given by the following proposition.

Proposition 32. *The exact completion induces a biequivalence between the 2-category of weakly lex category in which idempotents split and weakly lex functors between them and the 2-category of exact categories with enough projectives and exact functors which preserve projectives between them.*

As a consequence, we have that two exact categories with enough projectives *are equivalent if and only if they have two equivalent projective covers.*

3.4. Colimits in the exact completion

In our main examples of exact completions of weakly lex categories, that is monadic categories over (a power of) Set, we have the following situation: the weakly lex category $\mathbb{C}$ has small sums which are computed in $\mathbb{C}_{\mathrm{ex}}$ and $\mathbb{C}_{\mathrm{ex}}$ is cocomplete. This section is devoted to the study of this situation.

Lemma 33. *Let $\mathbb{A}$ be a category with weak kernel pairs and $\mathbb{P}$ be a projective cover of $\mathbb{A}$; the full inclusion $\mathbb{P} \to \mathbb{A}$ preserves the sums which turn out to exist in $\mathbb{P}$.*

Proof. We write the proof for a binary sum, but the argument is general. Consider a sum in $\mathbb{P}$

$$P_1 \xrightarrow{s_1} P \xleftarrow{s_2} P_2$$

and two arrows in $\mathbb{A}$

$$P_1 \xrightarrow{x_1} X \xleftarrow{x_2} P_2$$

with $\mathbb{P}$-cover $q: Q \to X$; we obtain extensions $y_1 : P_1 \to Q$ and $y_2 : P_2 \to Q$ such that $qy_1 = x_1$ and $qy_2 = x_2$. Since Q is in $\mathbb{P}$, there exists $y : P \to Q$ such that $ys_1 = y_1$ and $ys_2 = y_2$. Then $qy : P \to X$ is the required factorization.

Uniqueness: Suppose that $f, g : P \rightrightarrows X$ are two arrows such that $fs_1 = gs_1$ and $fs_2 = gs_2$. Consider two extensions $\bar{f} : P \to Q$ and $\bar{g} : P \to Q$ such that $q\bar{f} = f$ and $q\bar{g} = g$. Now, $q\bar{f}s_1 = fs_1 = gs_1 = q\bar{g}s_1$, so that there exists $t_1 : P_1 \to N(q)$ such that $q_0 t_1 = \bar{f}s_1$ and $q_1 t_1 = \bar{g}s_1$, where $q_0, q_1 : N(q) \rightrightarrows Q$ is a weak kernel pair of $q : Q \to X$. Similarly, there exists $t_2 : P_2 \to N(q)$ such that $q_0 t_2 = \bar{f}s_2$ and $q_1 t_2 = \bar{g}s_2$. Now, from the first part of the proof, we obtain $t : P \to N(q)$ such that $ts_1 = t_1$ and $ts_2 = t_2$. Moreover, $\bar{f}s_1 = q_0 t_1 = q_0 ts_1$ and $\bar{f}s_2 = q_0 t_2 = q_0 ts_2$, so that $\bar{f} = q_0 t$ because Q is in $\mathbb{P}$; similarly, $\bar{g} = q_1 t$, because $\bar{g}s_1 = q_1 t_1 = q_1 ts_1$ and $\bar{g}s_2 = q_1 t_2 = q_1 ts_2$. Finally, $f = q\bar{f} = qq_0 t = qq_1 t = q\bar{g} = g$. □

Corollary 34. *Let $\mathbb{C}$ be a weakly lex category; the embedding $\Gamma : \mathbb{C} \to \mathbb{C}_{ex}$ preserves the sums which turn out to exist in $\mathbb{C}$.*

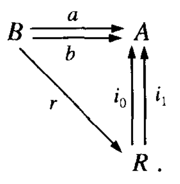
Let us fix some notations for the next lemma: if $\mathbb{A}$ is a category, $\theta(\mathbb{A})$ is the ordered reflection of $\mathbb{A}$; if A is an object of $\mathbb{A}$, $\mathrm{Sub}(A)$ is the ordered class of subobjects of A and $\mathbb{A}/A$ is the usual comma category.

Lemma 35. *Let $\mathbb{A}$ be a category with (strong epi)-(mono) factorization and $\mathbb{P}$ a strong projective cover of $\mathbb{A}$; for each object A of $\mathbb{A}$, $\mathrm{Sub}(A)$ and $\theta(\mathbb{P}/A)$ are isomorphic ordered classes.*

Proof. First, $\mathrm{Sub}(A) \to \theta(\mathbb{P}/A)$: given a monomorphism $X \to A$, we can consider a $\mathbb{P}$-cover $P \to X$ of X and we obtain an element of $\theta(\mathbb{P}/A)$ taking the composite $P \to X \to A$; the order is preserved because the objects of $\mathbb{P}$ are strong projective. Second, $\theta(\mathbb{P}/A) \to \mathrm{Sub}(A)$: given an object $P \to A$ of $\mathbb{P}/A$, we can take the monic part of its factorization; the order is preserved because, by definition, strong epimorphisms are orthogonal to monomorphisms. Clearly, $\mathrm{Sub}(A) \to \theta(\mathbb{P}/A)$ and $\theta(\mathbb{P}/A) \to \mathrm{Sub}(A)$ are inverses. □

Proposition 36. *Let $\mathbb{C}$ be a weakly lex category; if $\mathbb{C}$ has sums and $\mathbb{C}_{ex}$ is well-powered, then $\mathbb{C}_{ex}$ is cocomplete.*

Proof. First, the coequalizers: Consider two arrows in $\mathbb{C}_{ex}$ with their (regular epi)-(jointly monic) factorization



Now we can consider the equivalence relation $j_0, j_1 : E \rightrightarrows A$ generated by the relation $i_0, i_1 : R \rightrightarrows A$, that is the intersection of all the equivalence relations in A which contain $i_0, i_1 : R \rightrightarrows A$. This intersection exists: by the previous lemma, $\mathrm{Sub}(A)$ is isomorphic to

$\theta(\mathbb{P}/A)$ which is cocomplete because $\mathbb{P}$ has sums; so Sub(A) is also complete. Since $\mathbb{C}_{ex}$ is exact, $j_0, j_1 : E \rightrightarrows A$ has a coequalizer which is clearly also the coequalizer of $i_0, i_1 : R \rightrightarrows A$ and then of $a, b : B \rightrightarrows A$.

Sums: Once again we sketch the proof for a binary sum, but it is general. Consider two objects $r_0, r_1 : R \rightrightarrows X$ and $s_0, s_1 : S \rightrightarrows Y$ of $\mathbb{C}_{ex}$ and the coequalizer

$$\Gamma(R \amalg S) \underset{\Gamma(r_1 \amalg s_1)}{\overset{\Gamma(r_0 \amalg s_0)}{\rightrightarrows}} \Gamma(X \amalg Y) \to Q,$$

which exists from the first part of the proof. Using Corollary 34 and Proposition 15, an interchange argument shows that Q is the sum of the two given objects. □

Coming back to Lemma 35, let us observe that, given an object A of $\mathbb{A}$ and a $\mathbb{P}$-cover $p : P \to A$, the composition with p gives a surjection from the object of $\mathbb{P}/P$ to the object of $\mathbb{P}/A$. Using the axiom of choice, we can therefore inject the object of $\mathbb{P}/A$ in $\mathbb{P}/P$. Moreover, if two objects of $\mathbb{P}/A$ are identified in $\theta(\mathbb{P}/P)$, then they are identified also in $\theta(\mathbb{P}/A)$. So, by virtue of Lemma 35, we have proved the following lemma.

Lemma 37. *Let $\mathbb{A}$ be a category with (strong epi)-(mono) factorization and $\mathbb{P}$ a strong projective cover of $\mathbb{A}$; $\mathbb{A}$ is well-powered if and only if for each P in $\mathbb{P}$, $\theta(\mathbb{P}/P)$ is a small set.*

Corollary 38. *Let $\mathbb{C}$ be a weakly lex category; $\mathbb{C}_{ex}$ is well-powered if and only if, for each X in $\mathbb{C}$, $\theta(\mathbb{C}/X)$ is a small set.*

4. Examples and applications

4.1. Monadic categories

Let $\mathbb{T}$ be a monad over Set and write EM($\mathbb{T}$) and KL($\mathbb{T}$), respectively, for the Eilenberg–Moore and the Kleisli category of $\mathbb{T}$. It is well known that EM($\mathbb{T}$) is an *exact category* and that KL($\mathbb{T}$) is a *projective cover* of it, so that by the characterization theorem, EM($\mathbb{T}$) *is the exact completion of* KL($\mathbb{T}$). Hence, in particular, a categories of algebras is "*the exact completion of the category of free algebras*", quite a remarkable fact. This statement, in the full generality of the monadicity notion, tells us also, for instance, that "*the category of compact Hausdorff spaces is the exact completion of the category of extremally disconnected spaces*" (see e.g. [16] for the monadicity of compact Hausdorff spaces as well as for the projectives there).

Let us now show how the theory so far developed provides a very simple proof of a known characterization (see e.g. [2]) of categories monadic over Set (see [22] for more details). We begin with a simple lemma, whose proof is straightforward.

Lemma 39. *Let $\mathbb{C}$ be a category. The following conditions are equivalent:*

(1) *$\mathbb{C}$ is equivalent to the category* KL($\mathbb{T}$) *for a monad $\mathbb{T}$ over* Set;

(2) *there exists an object $G \in \mathbb{C}$ such that*

(i) *for each set I, the I-indexed copower $I \bullet G$ of G exists;*

(ii) *$\mathbb{C}$ is locally small, and for each object X of $\mathbb{C}$ there exists a set I such that $X \simeq I \bullet G$.*

Theorem 40. *Let $\mathbb{A}$ be a category. The following conditions are equivalent:*

(1) *$\mathbb{A}$ is equivalent to the category of algebras* EM($\mathbb{T}$) *for a monad $\mathbb{T}$ over* Set;

(2) *$\mathbb{A}$ is a locally small exact category with a* projective *regular generator, where regular generator means an object G such that*

(i) *all copowers of G exist;*

(ii) *for all objects A of $\mathbb{A}$ there exists a set I and a regular epi*

$$I \bullet G \to A.$$

Proof. The implication (1)$\Rightarrow$(2) being obvious, let us show the implication (2)$\Rightarrow$(1) : Let $\mathbb{C}$ be the full subcategory of $\mathbb{A}$ spanned by $I \bullet G$ for $I \in$ Set; $\mathbb{C}$ is a projective cover of $\mathbb{A}$. But $\mathbb{C}$ is equivalent to KL($\mathbb{T}$) and KL($\mathbb{T}$) is a projective cover of EM($\mathbb{T}$). □

Observe that the characterization theorem can easily be generalized to monadic categories over a power of Set, simply by requiring a *set* of projective regular generators.

In the general case where $\mathbb{E}$ is a topos, there are two related examples of monadic categories over $\mathbb{E}$, which in spite of the failure of the axiom of choice can be proved to be exact with enough projectives, and hence the exact completion of the projectives. One is the category of algebras for the covariant power set functor, i.e. the category SL($\mathbb{E}$) of sup-lattices in $\mathbb{E}$, whose full subcategory of projectives is the splitting of the idempotents of the category of relations of $\mathbb{E}$ (see [22] for a proof). This last, whose objects are also called "infosys", has been recently studied in [20], where, in terms of a sup-lattice property, the "*constructive complete distributivity*" has been characterized.

The other example is the category of algebras for the "double dual" monad on $\mathbb{E}$, which is well known to be the dual category $\mathbb{E}^{op}$ of the topos $\mathbb{E}$. Easily one can show that $\mathbb{E}^{op}$ is an exact category with enough projectives.

Finally, we should mention a new result obtained in a very simple way using the universal property of the exact completion; we refer to [23] for the complete details.

Theorem 41. *Let $\mathbb{A}$ be a category. The following conditions are equivalent:*

(1) *$\mathbb{A}$ is a localization (resp. an epireflective category, i.e. a reflective full subcategory with the unit of the reflection a regular epi) of a category of algebras* EM($\mathbb{T}$) *for a monad $\mathbb{T}$ over* Set;

(2) *$\mathbb{A}$ is a locally small exact category (resp. regular category with coequalizers of equivalence relations) with a (projective) regular generator.*

We mentioned the case of epireflective categories of monadic ones, although it is already known, because an interesting aspect of the proof contained in [23] is that the same proof gives at once the known case and the not known one. Also this characterization theorem generalizes at the case of categories monadic over a power of Set.

4.2. Presheaf categories

We can use our theory to give a new proof of a well-known characterization of presheaf categories (cf. [5, 22]), which has the advantage to be appliable to the study of geometric morphisms of toposes, and that will be used in the next section.

Let $\mathbb{D}$ be a small category and Fam$\mathbb{D}$ its *sum-completion.* Fam$\mathbb{D}$ can be described as the category whose objects are families $(D_i)_{i\in I}$ of objects of $\mathbb{D}$ indexed by a set I, and whose arrows

$$\langle \underline{f}, \phi\rangle : (D_i)_{i\in I} \to (D_j)_{j\in J}$$

are given by a function $\phi : I \to J$ and a family

$$\underline{f} = (f_i : D_i \to D_{\phi(i)})_{i\in I}$$

of arrows of $\mathbb{D}$. $\mathbb{B} = \mathrm{Fam}\mathbb{D}$ is an (*infinitary*) *extensive category* (see [8]) in the sense that has small sums, and for all families $(X_i)_{i\in I}$ of objects, the functor sum

$$\prod \mathbb{B}/X_i \to \mathbb{B}/\coprod X_i$$

is an equivalence. Fam$\mathbb{D}$ is naturally equivalent to the full subcategory of $\mathscr{P}\mathbb{D}$ spanned by sums of representable presheaves. This implies that Fam$\mathbb{D}$ is a projective cover of $\mathscr{P}\mathbb{D}$.

Lemma 42. *Let $\mathbb{B}$ be a category. The following conditions are equivalent:*

(1) *$\mathbb{B}$ is equivalent to the category* Fam$\mathbb{D}$ *for a small category $\mathbb{D}$;*

(2) *$\mathbb{B}$ is locally small with sums, and there exists a small subcategory $\mathbb{D}$ of $\mathbb{B}$ consisting of* indecomposable *objects, i.e. of objects for which the covariant hom-functor preserves sums, such that each object X is isomorphic to a sum of indecomposables.*

Observe that in an extensive category, an object is indecomposable if and only if it is not initial and it cannot be decomposed as a sum of not initial objects.

Corollary 43. *Let $\mathbb{A}$ be a category. The following conditions are equivalent:*

(1) *$\mathbb{A}$ is equivalent to the category of presheaves on a small category;*

(2) *$\mathbb{A}$ is locally small exact extensive, and has a set $G = \{G_j\}_J$ of projective and indecomposable generators (here generators mean that maps out of them are enough to distinguish equality between arbitrary pairs of maps).*

Proof. Calling $\mathscr{G}$ the full subcategory of $\mathbb{A}$ determined by generators, then the presheaf category on $\mathscr{G}$ and the category $\mathbb{A}$ are exact categories with equivalent projective covers. For, since $\mathbb{A}$ is extensive and each object of $\mathscr{G}$ is indecomposable, the extension to Fam$\mathscr{G}$ of the inclusion of $\mathscr{G}$ in $\mathbb{A}$ is full and faithful; moreover, since each object of $\mathscr{G}$ is projective, Fam$\mathscr{G}$ is a full subcategory of projectives in $\mathbb{A}$; finally, one can show that Fam$\mathscr{G}$ is a projective cover of $\mathbb{A}$, using that the objects in $\mathscr{G}$ are generators, and that in any (finitary) extensive exact category every epi is regular (see [11, 1.652], for an elementary proof of the last sentence). □

Observe that any extensive exact category is in fact cocomplete, since using extensivity and exactness one can show that the usual way to construct coequalizers of parallel pairs f, g as the quotient of the equivalence relation generated by the image of the map $\langle f, g \rangle$, still works.

Observe also that a category satisfying condition (2) of the corollary is monadic over a power of Set, because by extensivity a set G of generators is also a set G of regular generators; in other words, the only two differences between categories monadic over a power of Set and presheaf categories are in the *extensivity* of the sums and in the *indecomposability* of generators.

Finally, let us observe that the extensivity and exactness conditions in Corollary 43 could be weakened using the existence of projective and indecomposable generators, so allowing a precise comparison with the known characterization of [5], but we do not enter here in this analysis.

4.3. Grothendieck toposes

Let us recall that a *Grothendieck topos* is a locally small category $\mathbb{A}$ satisfying one of the following three equivalent conditions:

(1) $\mathbb{A}$ satisfies Giraud axioms for a topos, that is $\mathbb{A}$ is an infinitary extensive exact category with a set of generators;

(2) $\mathbb{A}$ is a localization of a presheaf category;

(3) $\mathbb{A}$ is equivalent to the category of sheaves on a site.

Usually (see e.g. [4, 15, 18, 19]) the proof of the equivalence runs as follows. First, one proves that the associated sheaf functor exhibits a category of sheaves as a localization of the corresponding presheaf category (that is, $(3) \Rightarrow (2)$). Second, one observes that Giraud axioms are verified by a presheaf category and that they are stable under localizations (that is, $(2) \Rightarrow (1)$). Third, starting from the family of generators involved in Giraud axioms, one constructs a site and an equivalence between the given category and the resulting category of sheaves (that is, $(1) \Rightarrow (3)$), and this is the part which classically is called "Giraud theorem characterizing toposes". We think it is of some interest to have a direct proof that (1) implies (2), using that a presheaf category is an exact category with enough projectives, and hence using our theory of left covering functors.

For this, we look more carefully at the sum-completion Fam$\mathbb{D}$ of a small category $\mathbb{D}$. We know that Fam$\mathbb{D}$ is (equivalent to) a projective cover of $\mathscr{P}\mathbb{D}$, so that it is a weakly lex category. Let us describe explicitly some weak finite limits in Fam$\mathbb{D}$, identified with the full subcategory of $\mathscr{P}\mathbb{D}$ spanned by sums of representable functors. Each time, one can consider the corresponding honest limit in $\mathscr{P}\mathbb{D}$ and use the canonical presentation of a presheaf as colimit (that is quotient of a sum) of representable presheaves. We give directly the resulting formula.

A weak terminal object in Fam$\mathbb{D}$ is the coproduct $\coprod_X \mathbb{D}(-,X)$ of all the representable presheaves.

A weak product of two objects $\mathbb{D}(-,A)$ and $\mathbb{D}(-,B)$ in Fam$\mathbb{D}$ is the coproduct $\coprod \mathbb{D}(-,X)$ indexed over all the pairs of arrows $A \xleftarrow{u} X \xrightarrow{v} B$ in $\mathbb{D}$ with X varying in $\mathbb{D}_0$.

A weak equalizer of two parallel maps $u,v : \mathbb{D}(-,A) \rightrightarrows \mathbb{D}(-,B)$ in Fam$\mathbb{D}$ is the coproduct $\coprod \mathbb{D}(-,X)$ indexed over all the arrows $x : X \to A$ in $\mathbb{D}$ such that $ux = vx$ with X varying in $\mathbb{D}_0$.

So, we have described binary products and equalizers of objects and arrows of Fam$\mathbb{D}$ coming from $\mathbb{D}$ via the (Yoneda) embedding $\mathbb{D} \to$ Fam$\mathbb{D}$.

Lemma 44. *Let $\mathbb{D}$ be a small category and $\mathbb{A}$ be a left exact extensive ("lextensive") category; a sum-preserving functor F:*Fam$\mathbb{D} \to \mathbb{A}$ *which is left covering with respect to binary weak products and weak equalizers of objects and arrows of* Fam$\mathbb{D}$ *coming from $\mathbb{D}$, is in fact left covering with respect to all binary weak products and weak equalizers.*

Proof (Products). First of all observe that extensivity implies that the *distributive law* holds (see [8]): given an object X and a family of object (A_k) in a lextensive category, then the canonical map

$$\coprod_k (X \times A_k) \to X \times \coprod_k A_k$$

is invertible. Then observe that in Fam$\mathbb{D}$ the following form of distributivity with respect to weak products holds:

"for each choice of weak products $X \times_w A_k$, then the coproduct $\coprod_k (X \times_w A_k)$ is also a weak product $X \times_w \coprod_k A_k$",

where now X and A_i are families of objects of $\mathbb{D}$. The result then follows by applying the previous two observations and from the fact that a coproduct of strong epis is a strong epi.

Equalizers: the result follows first by observing that in a lextensive category, an equalizer of the diagram of the kind

$$f,g : \coprod_k X_k \rightrightarrows A$$

is a coproduct of the equalizers e_k of the components fi_k, gi_k; then by observing that in Fam$\mathbb{D}$ the following statement holds:

"for each choice of weak equalizers e_k of the components fi_k, gi_k, then the coproduct of the e_k is a weak equalizer of f, g". □

The next proposition should also be compared with the characterization of localizations of monadic categories over a power of Set: the only difference is the *extensivity* of the sums.

Proposition 45. *Let $\mathbb{A}$ be a locally small extensive exact category; if $\mathbb{A}$ admits a set $\{G_i\}_I$ of generators, then $\mathbb{A}$ is a localization of a presheaf category on a small category.*

Proof. Consider the full subcategory $\mathbb{D}$ of $\mathbb{A}$ whose objects are the generators G_i and call $F : \mathbb{D} \to \mathbb{A}$ the inclusion. Using that $\mathbb{A}$ is extensive exact, we already observed that in $\mathbb{A}$ every mono is regular, and hence that *every epi is regular*, so that every object is canonically a quotient of a sum of generators; it follows that the set $\{G_i\}_I$ is *dense*, i.e. that the functor $\mathbb{A}(F-,-) : \mathbb{A} \to \mathscr{P}\mathbb{D}$ is full and faithful. We can now factor the Yoneda embedding $y : \mathbb{D} \to \mathscr{P}\mathbb{D}$ as

$$\mathbb{D} \xrightarrow{\mathscr{I}} \mathrm{Fam}\mathbb{D} \xrightarrow{\Gamma} (\mathrm{Fam}\mathbb{D})_{\mathrm{ex}} \simeq \mathscr{P}\mathbb{D}.$$

We can now define a left adjoint $\hat{F}$ to $\mathbb{A}(F, -.-)$ in two steps: first, consider the left Kan extension $F' : \mathrm{Fam}\mathbb{D} \to \mathbb{A}$ of F along $\mathscr{I}$ (that is, the sum-preserving extension of F) and then take the left Kan extension $\hat{F}$ of F' along Γ. By Proposition 31 and Theorem 29, to prove that $\hat{F}$ is left exact, it suffices to prove that F' is left covering. Keeping in mind the description of weak limits in Fam$\mathbb{D}$, we have to look at the following three canonical factorizations:

$$\coprod G_i \to 1, \quad \coprod G_i \to C \times D, \quad \coprod G_i \to E_{u,v},$$

where C, D, u, v are objects and arrows in $\mathbb{D}$; the first coproduct is indexed by all the objects G_i in $\mathbb{D}$; the second coproduct is indexed by all the pairs of arrows $C \leftarrow G_i \to D$ in $\mathbb{D}$ with G_i varying in $\mathbb{D}$; the third coproduct is indexed by all the arrows $w : G_i \to C$ such that $uw = vw$ with G_i varying in $\mathbb{D}$; 1 is the terminal in $\mathbb{A}$; $C \times D$ is the product in $\mathbb{A}$; $E_{u,v} \to C \underset{v}{\overset{u}{\rightrightarrows}} D$ is the equalizer in $\mathbb{A}$. These three arrows are regular epis exactly because the family $\{G_i\}_I$ generates 1, $C \times D$ and $E_{u,v}$. By Proposition 27 and Lemma 44, we have that F' is left covering and the proof is complete. □

4.4. Geometric morphisms

In this section we revisit the study of geometric morphisms from a topos $\mathscr{E}$ to a topos of presheaves $\mathscr{P}\mathbb{D}$, using the matter developed in the last two sections. We essentially follow the terminology of [18].

Definition 46. Let $\mathbb{D}$ be a small category, $y : \mathbb{D} \to \mathscr{P}\mathbb{D}$ be the Yoneda embedding and $\mathbb{A}$ be a cocomplete left exact category; a functor $F : \mathbb{D} \to \mathbb{A}$ is flat if its left Kan extension $\tilde{F} : \mathscr{P}\mathbb{D} \to \mathbb{A}$ along y is left exact.

Observe that in the previous definition one can equivalently require that $\tilde{F}$ is exact; in fact $\tilde{F}$, being computed pointwise, has always right adjoint given by $\mathbb{A}(F-,-) : \mathbb{A} \to \mathscr{P}\mathbb{D}$.

Proposition 47. *With the notations of the previous definition and supposing $\mathbb{A}$ exact, we have that F is flat if and only if its sum-preserving extension $F' : \mathrm{Fam}\mathbb{D} \to \mathbb{A}$ is left covering.*

Proof. The "if" part follows from Proposition 31 and Theorem 29, and the "only if" part follows from Proposition 25. □

Definition 48. Let $\mathbb{D}$ be a small category and $\mathbb{A}$ a left exact category. A functor $F : \mathbb{D} \to \mathbb{A}$ is filtering if the following three conditions hold:

(1) the family of all maps $FX \to 1$ (with X varying in $\mathbb{D}_0$) is epimorphic,

(2) for each pair A, B in $\mathbb{D}_0$, the family of all maps

$$\langle Fu, Fv \rangle : FX \to FA \times FB$$

(with $A \xleftarrow{u} X \xrightarrow{v} B$ in $\mathbb{D}$ with X varying in $\mathbb{D}_0$) is epimorphic,

(3) for each pair $u, v : A \rightrightarrows B$ in $\mathbb{D}_1$, the family of all maps $FX \to E_{u,v}$ (induced via the equalizer $E_{u,v} \to FA \underset{Fv}{\overset{Fu}{\rightrightarrows}} FB$ by maps $w : X \to A$ in $\mathbb{D}$ such that $uw = vw$ with X varying in $\mathbb{D}_0$) is epimorphic.

Proposition 49. *Consider a functor $F : \mathbb{D} \to \mathbb{A}$ with $\mathbb{D}$ small and $\mathbb{A}$ an exact extensive category; let $F' : \mathrm{Fam}\mathbb{D} \to \mathbb{A}$ be the sum-preserving extension of F; then F is filtering if and only if F' is left covering.*

Proof. Recall that in such a category $\mathbb{A}$ every epimorphism is regular. Keeping in mind the description of weak limits in $\mathrm{Fam}\mathscr{D}$ given at the beginning of Section 4.3, the three conditions of the previous definition are equivalent, respectively, to the fact that F' is left covering with respect to the terminal object, products of objects coming from $\mathbb{D}$ and equalizers of arrows coming from $\mathbb{D}$. The result immediately follows from Lemma 44 and Proposition 27. □

From Propositions 47 and 49, we obtain an easy proof of the well-known characterization of geometric morphisms $\mathbb{A} \to \mathscr{P}\mathbb{D}$ in terms of filtering functors $\mathbb{D} \to \mathbb{A}$; our proof holds if $\mathbb{A}$ is an extensive exact locally small category, which is more general than a cocomplete elementary topos (cf. Theorem 1, p. 399 of [18]). We think that Proposition 49 could contribute to clarify the definition of filtering functor.

4.5. *The epireflective hull*

In order to cover more examples, we need an extension of our theory.

Definition 50. A "*completely regular category*" $\mathbb{A}$ is a category which is complete and regular and such that the following condition holds: if

$$(p_i : X_i \to Y_i)_I$$

is a family of regular epi's, then the unique map

$$\prod_I p_i : \prod_I X_i \to \prod_I Y_i$$

is again a regular epi.

The above condition is already known in the context of abelian categories as the condition $Ab4^*$ (see [13]). If the family is finite, the condition is of course redundant, but if the family is infinite, the condition cannot be deduced from the completeness and the regularity of the category: as pointed out to us by F. Borceux, localic toposes are usually not completely regular categories.

Let us consider now a category $\mathbb{C}$ with all *small* weak limits; we can construct $\mathbb{C}^{\infty}_{\text{reg}}$ as in Definition 7, but an object is now a small (but not necessarily finite) family of arrows $(f_i : X \to X_i)_I$. Clearly, $\mathbb{C}^{\infty}_{\text{reg}}$ is a complete and regular category. Moreover, using once again the fact that a regular epi is, up to isomorphisms, of the form

$$[1_X] : (f_i : X \to X_i)_I \to (g_j : X \to Y_j)_J,$$

it is quite obvious to prove that $\mathbb{C}^{\infty}_{\text{reg}}$ is a completely regular category. As in Proposition 9, each object of $\mathbb{C}^{\infty}_{\text{reg}}$ can be embedded in a product of projective objects, but this product can now be infinite. Also the universal property of the embedding $\Gamma : \mathbb{C} \to \mathbb{C}^{\infty}_{\text{reg}}$ needs a modification as follows:

"*for each completely regular category* $\mathbb{A}$, *the canonical embedding*

$$\Gamma : \mathbb{C} \to \mathbb{C}^{\infty}_{\text{reg}}$$

induces an equivalence between the category of functors $F : \mathbb{C} \to \mathbb{A}$ *which are left covering with respect to all small weak limits, and the category of exact continuous functors* $\hat{F} : \mathbb{C}^{\infty}_{\text{reg}} \to \mathbb{A}$".

The characterization (cf. Proposition 9) is a bit more subtle: the obvious thing is to require that the category $\mathbb{A}$ is completely regular. But in this case this additional assumption can be avoided and $\mathbb{A}$ can be taken as a complete and regular category, because the extension $\hat{F} : \mathbb{P}^{\infty}_{\text{reg}} \to \mathbb{A}$ of the full inclusion $F : \mathbb{P} \to \mathbb{A}$ can be built up directly, without passing through the universal property of $\Gamma : \mathbb{P} \to \mathbb{P}^{\infty}_{\text{reg}}$.

We now point out some elementary properties of the notion of completely regular category. First of all, an example: clearly, the category Set is completely regular, because in Set epi means surjective and, since limits are computed pointwise, also presheaf categories are completely regular.

Proposition 51. *Let $\mathbb{T}$ be a monad on a completely regular category $\mathbb{A}$; if T sends regular epis into regular epis, then* EM($\mathbb{T}$) *is completely regular.*

Proposition 52. *Let $i:\mathbb{A} \hookrightarrow \mathbb{B}$ be a reflective subcategory and let $r:\mathbb{B} \to \mathbb{A}$ be the reflector: if $\mathbb{B}$ is regular and r is an* "epireflector", *that is the units $\eta_B : B \to i(rB)$ are regular epis, then $i:\mathbb{A} \to \mathbb{B}$ preserves regular epis and $\mathbb{A}$ is regular.*

Corollary 53. *Let $i : \mathbb{A} \hookrightarrow \mathbb{B}$ be an epireflective subcategory of a completely regular category $\mathbb{B}$ (with enough projectives); $\mathbb{A}$ is completely regular (with enough projectives).*

In particular, this proves that the category of Stone spaces is completely regular with enough projectives: in fact, it is well known that it is an epireflective subcategory of the category $\mathscr{CH}$ of compact Hausdorff spaces, which is monadic over Set, the reflector being the *Stone space $\pi_0(X)$ of connected components of X.*

Since by a theorem of Gleason (see [16]) we know that projectives in $\mathscr{CH}$ are exactly the extremally disconnected spaces, which are contained in the category of Stone spaces, and since a standard argument shows that each Stone space can be embedded in a (eventually infinite) product of projective objects (the product of as many copies of the two-point discrete space as the points of the Stone space), we know that *the category of Stone spaces is the regular completion (in the infinitary sense) of the category of extremally disconnected spaces.*

The example of Stone spaces leads us to come back to the problem, discussed in Proposition 17, of the reflectivity of $\mathbb{C}_{\mathrm{reg}}$ as subcategory of $\mathbb{C}_{\mathrm{ex}}$, which we will assume to be replete. Our aim is to show that, under some assumptions on the size of $\mathbb{C}$, $\mathbb{C}_{\mathrm{reg}}^{\infty}$ is epireflective in $\mathbb{C}_{\mathrm{ex}}$, and that it is the *epireflective hull of* $\mathbb{C}$ in $\mathbb{C}_{\mathrm{ex}}$ (cf. [12]).

Lemma 54. *Let $\mathbb{C}$ be a weakly lex (complete) category; $\mathbb{C}_{\mathrm{reg}}$ ($\mathbb{C}_{\mathrm{reg}}^{\infty}$) is closed under subobjects in $\mathbb{C}_{\mathrm{ex}}$.*

Proof. Let $(f_i : X \to X_i)_I$ be an object of $\mathbb{C}_{\mathrm{reg}}$ and $x_0, x_1 : \overline{X} \rightrightarrows X$ its embedding in $\mathbb{C}_{\mathrm{ex}}$ (that is x_0, x_1 is a weak universal pair such that $f_i x_0 = f_i x_1$, for all $i \in I$); consider now a monomorphism in $\mathbb{C}_{\mathrm{ex}}$

$$\begin{array}{ccc} R & \xrightarrow{\ \overline{f}\ } & \overline{X} \\ {\scriptstyle r_0}\downarrow\downarrow{\scriptstyle r_1} & & {\scriptstyle x_0}\downarrow\downarrow{\scriptstyle x_1} \\ Y & \xrightarrow[\ f\ ]{} & X \end{array}$$

and the object $(f_i f : Y \to X_i)_I$ of $\mathbb{C}_{\mathrm{reg}}$ together with its embedding in $\mathbb{C}_{\mathrm{ex}}$ $y_0, y_1 : \overline{Y} \rightrightarrows Y$ (that is, y_0, y_1 is a weak universal pair such that $f_i f y_0 = f_i f y_1$, for all $i \in I$). It is easy to see that $r_0, r_1 : R \rightrightarrows Y$ is isomorphic to $y_0, y_1 : \overline{Y} \rightrightarrows Y$. □

Proposition 55. *Let* $\mathbb{C}$ *be a weakly complete category and suppose that* $\mathbb{C}_{ex}$ *is well-powered;* $\mathbb{C}_{reg}^{\infty}$ *is the epireflective hull of* $\mathbb{C}$ *in* $\mathbb{C}_{ex}$.

Proof. Since $\mathbb{C}$ is weakly complete, $\mathbb{C}_{ex}$ is complete and moreover it is completely regular. $\mathbb{C}_{reg}^{\infty}$ is then closed in $\mathbb{C}_{ex}$ under the formation of products (because the embedding Ker: $\mathbb{C}_{reg}^{\infty} \to \mathbb{C}_{ex}$ is continuous), and of intersections, which exist because $\mathbb{C}_{ex}$ is well powered. Hence, a standard argument shows that $\mathbb{E} = \mathbb{C}_{reg}^{\infty}$ is (epi)reflective in $\mathbb{A} = \mathbb{C}_{ex}$: for, since the reflectivity is equivalent to $\mathbb{E}$ having coequalizers of equivalence relations, let $R \rightrightarrows S$ be an equivalence relation in $\mathbb{E}$ and let $c : S \to C$ be the coequalizer in $\mathbb{A}$; define T as the intersection of all subobjects of C in $\mathbb{E}$ through which c factors; the factorization of c through T is the coequalizer in $\mathbb{E}$.

Using again Lemma 54 and the characterization of regular completions is now easy to see that $\mathbb{E}$ is the epireflective hull of $\mathbb{C}$. □

Further examples of infinitary regular completions are provided by a recent paper [3], where the authors point out that some other examples of monadic categories over sets, the dual category $\mathcal{GR}$ of grids, and the category $\mathcal{FR}$ of frames, have as epireflective subcategories two quite important categories, namely the dual category $\mathrm{Top}^{\mathrm{op}}$ of topological spaces, and the dual category $\mathrm{Sob}^{\mathrm{op}}$ of sober spaces. What we like to point out here, is that in fact more is true: they are not just epireflective, but in particular they are the *epireflective hull* of the projectives, as it can be easily shown. This in particular means that they are the infinitary *regular completions* of the categories of projectives. This is not for free, since, for instance, the category of torsion-free abelian groups is epireflective in abelian groups, but is not the epireflective hull of the projectives (rational numbers is a torison free, but cannot be embedded in a product of free abelian groups). This remark means that exact functors from these categories to completely regular categories are determined by their restriction to the subcategory of projectives.

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